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POLYNOMIAL IMAGES OF CIRCLES AND LINES

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1. Introduction. In Heins' book [1, p. 124], there is an elegant proof of the following assertion: *If γ_1 and γ_2 are circles in the complex plane with their centers at the origin, and if f is an entire function (i.e., a function analytic everywhere in the complex plane) such that $f(z) \in \gamma_2$ for infinitely many points $z \in \gamma_1$, then $f(\gamma_1) \subset \gamma_2$.* In this note, we prove that the same conclusion holds if f maps a certain finite number of points of γ_1 into γ_2 provided that f is a polynomial. Moreover, we permit γ_1 as well as γ_2 to be an arbitrary circle or straight line, and we determine the exact form that the polynomial f must assume in each extreme case. These results are then used to obtain some geometric properties of lemniscates.

Our proofs use only simple arithmetic properties of complex numbers and the fact that a polynomial with complex coefficients of degree n has at most n zeros.

2. Polynomial images of circles and lines. The following theorems are concerned with arbitrary circles and straight lines, but, for the sake of simplicity, we consider only the unit circle $\{z: |z| = 1\}$ and the real axis in their proofs.

THEOREM 1. *Let P be a polynomial with complex coefficients of degree n ($n \geqq 1$) that maps at least $2n+1$ points, counted according to multiplicity, of a circle $C_1 \equiv \{z: |z-c_1| = r_1\}$ ($r_1 > 0$, c_1 complex) into a circle $C_2 \equiv \{z: |z-c_2| = r_2\}$ ($r_2 > 0$, c_2 complex). Then $P(C_1) = C_2$, and P is of the form*

$$P(z) \equiv e^{i\alpha} r_2 \{(z - c_1)/r_1\}^n + c_2 \qquad (\alpha \text{ real}).$$

Preliminary remarks. The $2n+1$ points need not be distinct. If $z_1 \in C_1$, $w_2 \in C_2$, and z_1 is a zero of order m of the polynomial $P(z) - w_2$, then the phrase "counted according to multiplicity" means that z_1 is to be counted m times.

When we say that P maps z into a circle C_2, we mean, of course, that $P(z) \in C_2$ and not that $P(z)$ is inside C_2.

Proof. Consider the special case when $P(z) \equiv a_0 + a_1 z + \cdots + a_n z^n$ ($n \geqq 1$, $a_n \neq 0$) maps at least $2n+1$ points, counted according to multiplicity, of the unit circle into itself. If ζ is one of the points in question, then $|P(\zeta)| = 1$; consequently, $P(\zeta)\overline{P(\zeta)} = 1$ and

$$(a_0 + a_1\zeta + \cdots + a_n\zeta^n)(\bar{a}_0 + \bar{a}_1\bar{\zeta} + \cdots + \bar{a}_n\bar{\zeta}^n) = 1.$$

Multiplying both sides of the last equation by ζ^n, using the fact that $\zeta\bar{\zeta} = 1$, and performing some simple algebraic manipulations, we deduce that ζ is a root of the algebraic equation

$$\begin{aligned} (1) \qquad & a_n\bar{a}_0 z^{2n} + (a_n\bar{a}_1 + a_{n-1}\bar{a}_0) z^{2n-1} + \cdots \\ & + (a_n\bar{a}_k + a_{n-1}\bar{a}_{k-1} + \cdots + a_{n-k}\bar{a}_0) z^{2n-k} + \cdots \\ & + (a_n\bar{a}_n + \cdots + a_0\bar{a}_0 - 1) z^n + \cdots + a_0\bar{a}_n = 0. \end{aligned}$$

Since the polynomial equation (1) has at least $2n+1$ solutions, counted according to multiplicity, each coefficient on the left-hand side must be equal to 0.

From this one easily concludes that $a_0 = a_1 = \cdots = a_{n-1} = 0$ and $a_n\bar{a}_n = 1$, which completes the proof of the theorem. (In order to handle the case of multiple roots, it is necessary to verify that, if ζ is a point of the unit circle that is mapped into the unit circle with multiplicity m by P, then ζ is a k-fold root of (1) where $k \geqq m$.)

THEOREM 2. *A polynomial with complex coefficients of degree n ($n \geqq 1$) cannot map more than $2n$ points, counted according to multiplicity, of a circle into a line.*

Proof. Suppose that $P(z) \equiv a_0 + a_1 z + \cdots + a_n z^n$ ($n \geqq 1$, $a_n \neq 0$) maps at least $2n+1$ points, counted according to multiplicity, of the unit circle into the real axis. If ζ is one of the points in question, then $P(\zeta)$ is real, $P(\zeta) - \overline{P(\zeta)} = 0$, and

$$(a_0 + a_1\zeta + \cdots + a_n\zeta^n) - (\bar{a}_0 + \bar{a}_1\bar{\zeta} + \cdots + \bar{a}_n\bar{\zeta}^n) = 0.$$

Multiplying both sides of the last equation by ζ^n and using the fact that $\zeta\bar{\zeta} = 1$, we deduce that ζ is a root of

$$(2) \quad a_n z^{2n} + a_{n-1} z^{2n-1} + \cdots + a_1 z^{n+1} + (a_0 - \bar{a}_0) z^n + (-\bar{a}_1) z^{n-1} + \cdots + (-\bar{a}_n) = 0.$$

Since (2) has at least $2n+1$ solutions, counted according to multiplicity, the coefficients on the left-hand side must all be equal to 0, and this contradicts the hypothesis that P is not identically equal to a constant.

We might note here that, if in Theorem 2 we only require that P be an entire function, we can, in general, say nothing about how many times the image of an arbitrary circle will cut a straight line. However, if P is an entire function of finite order, Hellerstein and Korevaar [2] have found the best possible asymptotic bound for $\Phi(R)$ where $\Phi(R)$ is the number of points on $\{z: |z| = R\}$ where P is real.

THEOREM 3. *Let P be a polynomial with complex coefficients of degree n ($n \geqq 1$). If P maps at least $n+1$ points, counted according to multiplicity, of a line $L_1 \equiv \{at+b: -\infty < t < \infty\}$ (a, b complex) into a line $L_2 \equiv \{At+B: -\infty < t < \infty\}$ (A, B complex), then P maps L_1 into (onto) L_2 if n is even (odd), and P is of the form*

$$P(z) \equiv A\left\{q_0 + q_1\left(\frac{z-b}{a}\right) + \cdots + q_n\left(\frac{z-b}{a}\right)^n\right\} + B \qquad (q_0, q_1, \cdots, q_n \text{ real}).$$

Proof. Suppose that $P(z) \equiv a_0 + a_1 z + \cdots + a_n z^n$ ($n \geqq 1$, $a_n \neq 0$) maps at least $n+1$ points, counted according to multiplicity, of the real axis into the real axis. If x is one of the points in question, then $P(x)$ is real and $P(x) - \overline{P(x)} = 0$. Since x is real, this yields

$$(a_0 - \bar{a}_0) + (a_1 - \bar{a}_1)x + \cdots + (a_n - \bar{a}_n)x^n = 0.$$

Reasoning as above, we conclude that $a_k = \bar{a}_k$ ($k = 0, 1, \cdots, n$), that is, that the coefficients of P are all real.

THEOREM 4. *A polynomial with complex coefficients of degree n* ($n \geqq 1$) *cannot map more than* $2n$ *points, counted according to multiplicity, of a line into a circle.*

Proof. Suppose that $P(z) \equiv a_0 + a_1 z + \cdots + a_n z^n$ ($n \geqq 1$, $a_n \neq 0$) maps at least $2n+1$ points, counted according to multiplicity, of the real axis into the unit circle. If x is one of these points, then $|P(x)|^2 = 1$; that is,

$$(a_0 + a_1 x + \cdots + a_n x^n)(\bar{a}_0 + \bar{a}_1 x + \cdots + \bar{a}_n x^n) = 1.$$

This implies that $a_n \bar{a}_n = 0$, which is a contradiction.

3. Some geometric properties of lemniscates. Let P be a nonconstant polynomial with complex coefficients. If α is a positive real number, then

$$\{z: |P(z)| = \alpha\} \tag{3}$$

is called a lemniscate associated with P. In this connection, one also frequently (cf. [**5**], pp. 20–21) studies loci of the form

$$\{z: \arg [P(z)] = \phi \text{ or } \phi + \pi\}. \tag{4}$$

Theorems 1–4 obviously yield geometric properties of loci of the forms (3) and (4). We now state two of these results.

THEOREM 5. *Let P be a polynomial with complex coefficients of degree n* ($n \geqq 1$) *that has at least two distinct zeros. Then no lemniscate associated with P meets a circle in more than* $2n$ *points, counted according to multiplicity, where multiplicity is defined in the obvious way.*

THEOREM 6. *Let P be a polynomial with complex coefficients of degree n* ($n \geqq 1$), *Then no lemniscate associated with P meets a straight line in more than* $2n$ *points. counted according to multiplicity.*

Since, for small positive values of α, $\{z: |P(z)| = \alpha\}$ consists of small Jordan curves, one about each distinct zero of P (cf. [**5**], p. 19), it is obvious that Theorems 5 and 6, as well as Theorems 1 and 4, are sharp.

4. Conclusion. To see that Theorems 2 and 3 are sharp, consider $P(z) \equiv z^n$ and $Q(z) \equiv i(z-i)(z-1)(z-2) \cdots (z-n+1)$. P maps $e^{ik\pi/n}$ ($k = 0, 1, \cdots, 2n-1$) into the real axis; and Q maps the points $0, 1, \cdots, n-1$ into the real axis, but the coefficients of Q are not all real.

It is possible to give more sophisticated proofs of Theorems 1–4. We now sketch some of these proofs. In the proof of Theorem 1, one can observe that $R(z) \equiv P(z)\overline{P(1/\bar{z})}$ is a rational function of degree at most $2n$ that is equal to $|P(z)|^2$ if $|z| = 1$; since $R(z) = 1$ for $2n+1$ values of z, it follows that $R(z) \equiv 1$ and hence that P has no nonzero zero, which, in turn, implies that $P(z) \equiv a_n z^n (|a_n| = 1)$. (Cf. [**1**], p. 124 and [**4**], pp. 221–222.) Since the imaginary part of $P(e^{i\theta})$ is a trigonometric polynomial in the proof of Theorem 2, the desired conclusion follows, as has been pointed out to the authors by S. Hellerstein, from the fact that a trigonometric polynomial of degree n has at most $2n$ zeros in the interval $[0, 2\pi)$. (As in the proofs of Theorems 1–4, slightly more care must be taken in the case of multiple roots (cf. [3], pp. 53–55). Theorem 3 can be proved by using the Lagrange interpolation formula, or, alternatively, by using Cramer's rule to determine the coefficients of the polynomials. (As in the proofs of Theorems 1–4, slightly more care must be taken in the case of multiple roots (cf. [3], pp. 356–364).)

The methods of this note also apply to rational functions.

Perhaps a few words about the connection between algebraic geometry and the results of this note are in order. It is obvious (cf. [5], p. 19) that a lemniscate associated with a polynomial of degree n is an algebraic curve of degree $2n$, that is, is the locus of an equation of the form

$$\sum_{j=0}^{j+k\leq 2n}\sum_{k=0} a_{jk}x^j y^k = 0,$$

where the coefficients a_{jk} are real. Likewise, the locus (4) is an algebraic curve of degree n. According to Bézout's Theorem, a line meets the lemniscate in $2n$ points (real or imaginary, distinct or coincident) and a circle meets the lemniscate in $4n$ points, except for certain degenerate cases. From observations such as these, one can prove weak forms of each of the Theorems 1–6. (It does not seem worthwhile at this time to analyze the connection between "multiplicity" as used in this note and as used in the theory of algebraic curves.) Our theorems, of course, deal with real points of intersection. Moreover, if one takes advantage of the fact that the lemniscate is an n-circular $(2n)$-*ic* (cf. [5], p. 19), one can sharpen some of the results mentioned above.

The results of this note give rise to a number of geometric questions. For example, from Theorem 2 it follows that the image of a circle under a nonconstant polynomial of degree n cannot meet the boundary of a square in more than $8n$ points. Is the bound $8n$ sharp? If P_1 and P_2 are polynomials and C_1 and C_2 are circles, what can one say about the intersections of the image curves $P_1(C_1)$ and $P_2(C_2)$? If P_1 is of degree 1, the answer is given by Theorem 1.

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NEWTONIAN ANALOGUES OF THE TRIGONOMETRIC AND EXPONENTIAL FUNCTIONS

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The functions treated in this note are defined and studied by means of series. It is shown in the last section of the note that they are expressible in terms of trigonometric and exponential functions. However, it is the belief of the author that this does not cause the study to be without interest. In fact it affords a new approach to the trigonometric and exponential functions.

1. Definitions and convergence. Let $z^{(k)} = z(z-1) \cdots (z-k+1)$. Consider the two series

$$z - \frac{z^{(3)}}{3!} + \frac{z^{(5)}}{5!} + \cdots + (-1)^{n-1} \frac{z^{(2n-1)}}{(2n-1)!} + \cdots, \tag{1}$$

$$1 - \frac{z^{(2)}}{2!} + \frac{z^{(4)}}{4!} + \cdots + (-1)^{n-1} \frac{z^{(2n-2)}}{(2n-2)!} + \cdots. \tag{2}$$

These are Newton series. It is not difficult to treat convergence directly. However, the convergence of most Newton series is most easily handled by considering the associated Dirichlet series as described in the following theorem. (See, for example, [1, p. 175].)

THEOREM I. *The series*

$$\sum_{k=1}^{\infty} c_k \frac{z^{(k)}}{k!} \tag{3}$$

and the series

$$\sum_{k=1}^{\infty} c_k \frac{(-1)^k}{k^{z+1}} \tag{4}$$

converge and diverge at the same points with the possible exception of 0, 1, 2, 3, $\cdots$. *If* (4) *converges uniformly over a bounded region, S, then so does* (3). *If* (3) *converges uniformly over S, the points,* 0, 1, 2, 3 $\cdots$ *being neither within nor on its boundary then so does* (4). *If one series converges absolutely at any point, with the possible exception of* 0, 1, 2, 3, $\cdots$, *then so does the other.*

THEOREM II. *Series* (1) *and* (2) *converge absolutely if* $R(z) > 0$. *They converge when* $z = 0$. *They converge uniformly over any bounded portion of a sectorial region with vertex at the origin and angle* θ *such that* $|\theta| \leqq \pi/2 - \delta$, $\delta > 0$.

Proof. The facts of this theorem are readily established by considering the Dirichlet series.

We write

$$\text{sint}\, z = z - \frac{z^{(3)}}{3!} + \frac{z^{(5)}}{5!} - \cdots + (-1)^{n-1} \frac{z^{(2n-1)}}{(2n-1)!} + \cdots, \tag{5}$$

$$\text{cost}\, z = 1 - \frac{z^{(2)}}{2!} + \frac{z^{(4)}}{4!} - \cdots + (-1)^{n-1}\frac{z^{(2n-2)}}{(2n-2)!} + \cdots. \tag{6}$$

2. Addition formulas

THEOREM III. *If* $R(x) > 0$ *and* $R(y) > 0$, *then*

$$\text{sint}(x+y) = \text{sint}\, x \,\text{cost}\, y + \text{cost}\, x \,\text{sint}\, y, \tag{7}$$

$$\text{cost}(x+y) = \text{cost}\, x \,\text{cost}\, y - \text{sint}\, x \,\text{sint}\, y. \tag{8}$$

Proof. It is known and can be readily proved that if $x \geqq 0$, $y \geqq 0$ then

$$(x+y)^{(n)} = x^{(n)} + nx^{(n-1)}y + \cdots + \binom{n}{r} x^{(n-r)}y^{(r)} + \cdots + y^{(n)} \tag{9}$$

Since, for example, $(-y)^{(2)} \neq y^{(2)}$, relations as $(x-y)^{(2)} = x^{(2)} - 2x^{(1)}y^{(1)} + y^{(2)}$ are not in general true. The coefficients in (9) are the coefficients in the ordinary binomial expansion. The proof is easily given by Newton's Interpolation Formula which is

$$f(x+y) = \frac{f(x)}{0!} + \frac{f(x)}{1!} y^{(1)} + \frac{\Delta^2 f(x)}{2!} y^{(2)} + \cdots + \frac{\Delta^{(n-1)} f(x)}{(n-1)!} y^{(n-1)} + R_n.$$

Here

$$R_n = \frac{x^{(n)}}{n!} \cdot \frac{d^n}{dx^n} f(\xi),$$

where ξ is greater than zero and smaller than the larger of the two numbers x, n.

Corresponding formulas to (7) and (8) involving the ordinary trigonometric functions can be proved by taking power series for $\sin x$, $\cos x$, $\sin y$ and $\cos y$ multiplying together, adding or subtracting and verifying the formulas. (See, for example, [2] p. 110). In the present case it is only necessary to copy the trigonometric work, putting a parenthesis about each exponent. It is necessary to notice that the manipulation of series, as is done, is under the assumption that the series are absolutely convergent. We consequently have formulas which hold when $x \geqq 0$ and $y \geqq 0$. But sint x and cost x are both analytic functions over a region of uniform convergence. Consequently formulas (7) and (8) are valid for values of x and y such that $R(x) > 0$ and $R(y) > 0$ and when x and/or y is zero. Formulas similar to the $\sin(x-y)$ formula simply do not hold in general. As a counterexample consider $\text{sint}(4-2) = \text{sint}\, 2 = 2$ but $\text{sint}\, 4\, \text{cost}\, 2 - \text{cost}\, 4\, \text{sint}\, 2 = 8$.

3. Difference relationships. Since $\Delta z^{(k)} = kz^{(k-1)}$ we see that

$$\Delta\, \text{sint}\, z = \text{cost}\, z, \tag{10}$$

$$\Delta\, \text{cost}\, z = -\,\text{sint}\, z. \tag{11}$$

Both functions satisfy the difference equation

$$\Delta^2 y(z) + y(z) = 0. \tag{12}$$

We next consider the difference Wronskian,

$$W = \begin{vmatrix} \Delta \operatorname{sint} z & \Delta \operatorname{cost} z \\ \operatorname{sint} z & \operatorname{cost} z \end{vmatrix}. \tag{13}$$

We readily verify that W satisfies the difference equation

$$\Delta W(z) = W(z) \tag{14}$$

or

$$W(z+1) = 2W(z). \tag{15}$$

By means of the addition formulas, and the defining series, we also establish the relations

$$\operatorname{sint}(x+2) = 2 \operatorname{cost} x, \tag{16}$$

$$\operatorname{cost}(x+2) = -2 \operatorname{sint} x, \tag{17}$$

$$\operatorname{sint}(x+4) = -4 \operatorname{sint} x, \tag{18}$$

$$\operatorname{cost}(x+4) = -4 \operatorname{cost} x. \tag{19}$$

These relations immediately show that there is no real period.

A function which satisfies a relation such as (18) is sometimes called *periodic of the second kind with multiplier.*

4. Real zeros.

THEOREM IV. *The smallest positive zero of cost x is* 2 *and of sint x is* 4.

Proof.

$$\operatorname{cost} x = 1 + \left(-\frac{x^{(2)}}{2!} + \frac{x^{(4)}}{4!}\right) + \left(-\frac{x^{(6)}}{6!} + \frac{x^{(8)}}{8!}\right) + \cdots.$$

If $0<x<1$ the quantity inside each parenthesis is positive. Also cost $0=1$, cost $1=1$. Hence cost $x>0$ when $0\leqq x\leqq 1$. We now write

$$\operatorname{cost} x = \left(1 - \frac{x^{(2)}}{2!}\right) + \left(\frac{x^{(4)}}{4!} - \frac{x^{(6)}}{6!}\right) + \cdots.$$

If $1<x<2$ the quantity inside each parenthesis is positive. Hence cost $x>0$ $x<0$ when $0\leqq x<2$. It is immediate that cost $2=0$.

Similar reasoning shows that the smallest positive zero of sint x is 4. By (18) sint $4n=0$, by (19) $\operatorname{cost}(2+4n)=0$, $n=1, 2, 3, \cdots$. There are no other real zeros.

Now consider W, the difference Wronskian of sint x and cost x: $W=\operatorname{sint}^2 x + \operatorname{cost}^2 x$. It is immediate that $W>1$ when $0\leqq x\leqq 1$. Formula (15) then shows that $W>1$ for all real x and becomes infinite with x. We shall later show that $W(z)\neq 0$ when z is any complex number.

5. Linear independence.

THEOREM V. *The functions sint z and cost z are linearly independent.*

By this we mean that there do not exist two functions $C_1(z)$ and $C_2(z)$ with period 1 such that

$$C_1(z) \operatorname{sint} z + C_2(z) \operatorname{cost} z = 0$$

wherever sint z and cost z are defined, and where $C_1(z)$ and $C_2(z)$ are not both zero at any one point.

If $C_1(z)$ sint $z+C_2(z)$ cost $z=0$ then $C_1(z)$ and $C_2(z)$ are analytic but for a multiplier. We take this multiplier to be 1. Now a necessary and sufficient condition [**3**, p. 119] that sint x and cost x be linearly dependent with constant multipliers over any set of congruent points, $a, a+1, a+2, \cdots$ is that $W(a+n)=0$, $n=0, 1, 2, \cdots$. But $W\neq 0$ at any point on the real axis. Consequently sint z and cost z are linearly independent on the real axis. As a matter of fact a may be any point, real or complex. Consequently sint z and cost z are linearly independent over the right-hand half-plane and $W(z)\neq 0$ if $R(z)>0$.

If y is any solution of (12) then

$$y = c_1(z) \operatorname{sint} z + c_2(z) \operatorname{cost}(z), \tag{20}$$

where $c_1(z)$ and $c_2(z)$ have the period 1. This follows from the general theory of linear difference equations.

6. Differentiation.

THEOREM VI. *If $h>0$, then*

$$\lim_0 \frac{\operatorname{sint} h}{h} = \frac{\pi}{4} \quad \text{and} \quad \lim_0 \frac{\operatorname{cost} h - 1}{h} = (1/2) \ln 2. \tag{21}$$

Proof. Since $h>0$,

$$\frac{\operatorname{sint} h}{h} = 1 - \frac{(h-1)(h-2)}{3!} + \frac{(h-1)(h-2)(h-3)(h-4)}{5!} + \cdots. \tag{22}$$

This is a Newton series and has an associated Dirichlet series

$$1 - \frac{1}{3}\cdot\frac{1}{3^h} + \frac{1}{5}\,\frac{1}{5^h} + \cdots (-1)^{n-1}\frac{1}{(2n-1)}\cdot\frac{1}{(2n-1)^h} + \cdots$$

This series converges when $h=0$. It consequently converges uniformly over any sector with vertex at the origin and opening to the right. Consequently (22) converges uniformly over any bounded portion of this sector. Consequently at the origin the limit of the series is equal to the series of the limits. Hence

$$\lim_0 \frac{\operatorname{sint} h}{h} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots + (-1)^{n-1}\frac{1}{2n-1} + \cdots = \frac{\pi}{4}\cdot$$

Similarly

$$\lim_0 \frac{\operatorname{cost} h - 1}{h} = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \cdots + (-1)^{n-1}\frac{1}{2n} + \cdots = \frac{1}{2}\ln 2.$$

(If we wish we can write A instead of $\pi/4$ and B instead of $(1/2)\ln 2$. This is under the assumption that we do not know the sums of the two series.)

THEOREM VII.

$$\frac{d}{dz}\operatorname{sint} z = \frac{1}{2}\ln 2 \operatorname{sint} z + \frac{\pi}{4}\operatorname{cost} z \tag{23}$$

$$\frac{d}{dz}\operatorname{cost} z = -\frac{\pi}{4}\operatorname{sint} z + \frac{1}{2}\ln 2 \operatorname{cost} z. \tag{24}$$

Proof.

$$\frac{\operatorname{sint}(z+h) - \operatorname{sint} z}{h} = \frac{\operatorname{sint} z \operatorname{cost} h + \operatorname{cost} z \operatorname{sint} h - \operatorname{sint} z}{h}$$

$$= \operatorname{sint} z \frac{\operatorname{cost} h - 1}{h} + \operatorname{cost} z \frac{\operatorname{sint} h}{h} \cdot$$

The limit of this with $h>0$ is the expression given in (23). Now we have repeatedly pointed out that sint z and cost z are analytic functions of a complex variable. The derivative does not depend upon h being positive. We consequently have formula (23). Formula (24) is derived in like manner.

We find the second derivatives. By differentiating (23) and (24) we find

$$\frac{d^2 \operatorname{sint} z}{dz^2} = \left[\frac{1}{4}(\ln 2)^2 - \frac{\pi^2}{16}\right]\operatorname{sint} z + \left(\frac{\pi}{4}\ln 2\right)\operatorname{cost} z$$

$$\frac{d^2 \operatorname{cost} z}{dz^2} = \left(-\frac{\pi}{4}\ln 2\right)\operatorname{sint} z + \left[-\frac{\pi^2}{16} + \left(\frac{1}{2}\ln 2\right)^2\right]\operatorname{cost} z.$$

By mathematical induction

$$\frac{d^n \operatorname{sint} z}{dz^n} = A \operatorname{sint} z + B \operatorname{cost} z$$

$$A = \left(\frac{\ln 2}{2}\right)^n - \binom{n}{2}\left(\frac{\ln 2}{2}\right)^{n-2}\left(\frac{\pi}{4}\right)^2 + \cdots$$

$$+ (-1)^{j-1}\binom{n}{2j}\left(\frac{\ln 2}{2}\right)^{n-2j}\left(\frac{\pi}{4}\right)^{2j} + \cdots, \quad 2j \leqq n,$$

$$B = \binom{n}{1}\left(\frac{\ln 2}{2}\right)^{n-1}\frac{\pi}{4} + \cdots$$

$$+ (-1)^{j-1}\binom{n}{2j-1}\left(\frac{\ln 2}{2}\right)^{n-(2j-1)}\left(\frac{\pi}{4}\right)^{2j-1} + \cdots$$

$$\frac{d^n \operatorname{cost} z}{dz^n} = A \operatorname{cost} z - B \operatorname{sint} z.$$

7. Graphs. We define a new function,

$$\text{tant } x = \frac{\text{sint } x}{\text{cost } x}, \quad \text{cost } x \neq 0.$$

By using (23) and (24) we find

$$\frac{d}{dx}\,\text{tant } x = \left(\frac{\pi}{4}\right)\frac{\text{sint}^2 x + \text{cost}^2 x}{\text{cost}^2 x}\,.$$

We have seen that $\text{sint}^2 x + \text{cost}^2 x > 0$. Consequently tant x increases with x wherever defined. Now consider sint x over the interval $0 < x < 2$. Over this interval d/dx sint $x > 0$ since the smallest positive zero of cost x is 2 and of sint x is 4. The graph consequently rises over this interval. There is a point of inflection where

$$\text{tant } x = \frac{\frac{\pi}{4}\ln 2}{\frac{\pi^2}{16} - \frac{1}{4}(\ln 2)^2}\,.$$

This point can be approximated. If we denote it by x_0 the curve is concave upward when $0 < x < x_0$ and convex upward when $x_0 < x < 2$. We know that sint $0 = 0$, sint $1 = 1$, and sint $2 = 2$.

Now consider the interval $2 \leqq x \leqq 4$. The curve is convex upward over this interval since d^2 sint x/dx^2 is negative over it. Now sint 2 = sint 3. Consequently there is a maximum between 2 and 3. There can only be one on account of the convexity of the curve. There are no further maxima or minima or points of inflection on the interval $0 < x \leqq 4$ inasmuch as the first derivative has but one zero on this interval and the second but one. Both are continuous. A detailed analysis of the graph of cost x is not made. An interested reader can easily make this analysis. Both graphs can be extended beyond the interval [0, 4] by means of (18) and (19).

8. The exponentialt. Consider

$$\text{(25)} \qquad \text{expt } z = 1 + z + \frac{z^{(2)}}{2!} + \cdots + \frac{z^{(n-1)}}{(n-1)!} + \cdots$$

The associated Dirichlet series converges when $z = 0$. Hence this series converges uniformly over any bounded portion of the sector with origin as vertex and opening to the right, $0 \leqq |\theta| \leqq \pi/2 - \delta$, $\delta > 0$. It is immediate that

$$\text{(26)} \qquad \Delta \text{ expt } z = \text{expt } z,$$

or

$$\text{(27)} \qquad \text{expt } (z + 1) - 2 \text{ expt } z = 0.$$

If we use Newton's formula and (14) we readily prove that $W(z) = \text{expt } z$. (See Section 10.)

THEOREM VIII.

$$\text{expt}\,(x+y) = (\text{expt}\,x)(\text{expt}\,y). \tag{28}$$

Proof. This formula can be established (see, for example, [2] p. 120) by using series exactly as the addition formulas for sint z and cost z were proved. The inverse of the exponentialt will be denoted by logt z. The exponentialt is readily calculated for positive integral values of the argument. This gives us certain values for logt x. Other values for real x can be calculated approximately by the Lagrange or other interpolation formula. We can show that

$$\frac{d}{dz}\,\text{expt}\,z = \ln 2\ \text{expt}\,z$$

and also that

$$\frac{d}{dz}\,\text{logt}\,z = \frac{1}{z\ln 2},\quad R(z) > 0.$$

9. Euler Forms. We now give extended definitions of our functions. Let

$$z(z-k)(z-2k)\cdots(z-(n-1)k) = \overset{(n)}{z_k},\quad \overset{(0)}{z_k} = 1$$

$$\text{sint}_k\,z = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\overset{(2n-1)}{z_k}}{(2n-1)!} \tag{29}$$

$$\text{cost}_k\,z = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\overset{2n-2}{z_k}}{(2n-2)!} \tag{30}$$

$$\text{expt}_k\,z = \sum_{n=1}^{\infty}\frac{\overset{(n-1)}{z_k}}{(n-1)!} \tag{31}$$

These series are all absolutely convergent if $k \leqq 1$ and $R(z/k) > 0$. For example compare the general term of (29) with the general term of the series for sint z/k. Granting these facts we readily verify the following relations: If $R(z)>0$, $R(iz)>0$, then

$$\text{expt}\,zi = \text{cost}_{1/i}\,z + i\,\text{sint}_{1/i}\,z \quad\text{and}\quad \text{expt}_{1/i}\,z = \text{cost}\,zi - i\,\text{sint}\,zi.$$

10. Trigonometric and exponential functions. To the present point in this paper we have worked as if the trigonometric and exponential functions were unknown. However, as a matter of fact we have the following theorem.

THEOREM IX.

$$\text{expt}\,z = 2^z \tag{32}$$

$$\text{sint}\,z = 2^{z/2}\sin\frac{\pi}{4}z \tag{33}$$

$$\text{cost } z = 2^{z/2} \cos \frac{\pi}{4} z. \tag{34}$$

Apply Leibnitz rule for the differentiation of a product to $2^{z/2} \sin (\pi/4)z$. We obtain

$$(d^n/dz^n)2^{z/2} \sin (\pi/4)z = A2^{z/2} \sin (\pi/4)z + B2^{z/2} \cos (\pi/4)z$$

where A and B are precisely the same as that given in Section 6. Now sint z and $2^{z/2}$ sint $\pi/4z$ are analytic functions when $0 < R(z) < \infty$. All their derivatives are the same at a point, for example $z=4$. Hence the functions are identical when $0 < R(z) < \infty$. Similar reasoning can be applied to cost z and to expt z.

With 2^z given by (32), formulas (33) and (34) serve to define sin z and cos z. The functions sint z, cost z, expt z can be analytically continued to the left of the axis of imaginaries by means of their addition formulas.

It is to be noticed that if the approach to the elementary transcendental functions is that which we have given, namely, by Newton's formula with difference interval 1, then 2 and not "e" is the fundamental number and $\log_2 x$ is the "natural" logarithm.

This paper is an outgrowth of a project undertaken with the support of the National Aeronautics and Space Administration.

References

1. T. Fort, Infinite Series, Clarendon, Oxford, 1930.
2. T. Fort, Trigonometry and the Elementary Transcendental Functions, Macmillan, New York, 1963.
3 T. Fort, Finite Differences and Difference Equations in the Real Plane, Clarendon, Oxford, 1948.

PASCAL MATRICES

DANIEL A. MORAN, Michigan State University

1. Introduction. By a *Pascal matrix* we mean an infinite triangular matrix $\|a_{ij}\|$ whose entries satisfy

$$a_{ij} = a_{i,j-1} + a_{i-1,j-1} \quad (i < j). \tag{1}$$

Thus such a matrix is fully determined when its top row and its main diagonal are specified. (If each entry in these positions is the integer 1, the reader will recognize the familiar "Pascal Triangle" of binomial coefficients). We shall show how to use these matrices to solve the binomial-recurrence difference equation (1), with arbitrary initial conditions, in any abelian group. This will lead to quick derivations of a class of identities involving sums of binomial coefficients.

For historical and technical reasons, we shall call the top row of an infinite matrix its 0th *row* and its extreme left-hand column its 0th *column*. A Pascal matrix, then, will be written

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For historical and technical reasons, we shall call the top row of an infinite matrix its 0th *row* and its extreme left-hand column its 0th *column*. A Pascal matrix, then, will be written

$$\begin{Vmatrix} a_{00} & a_{01} & a_{02} & \cdots \\ 0 & a_{11} & a_{12} & \cdots \\ 0 & 0 & a_{22} & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{Vmatrix}.$$

Since (1) imposes no condition on a_{00}, we shall assume that $a_{00}=0$ in all that follows. With this restriction, each Pascal matrix with entries from an abelian group G represents the complete solution of the binomial recurrence (1) for some set of initial values $[a_{01}, a_{02}, a_{03}, \cdots; a_{11}, a_{22}, a_{33}, \cdots]$. Conversely, each such set of initial values corresponds to a Pascal matrix, via the simple process of recursively constructing the triangular array for the given values.

2. A generating set of Pascal matrices. Let $A^{(n)}$ denote the Pascal matrix corresponding to the set $[a_{0i}; a_{ii}]$ of integral initial values, each of which is 0 except for $a_{0n}=1$; let $B^{(n)}$ denote the Pascal matrix whose initial values are all 0 except for $a_{nn}=1$. The reader will easily verify that

$$A^{(n)} = \|a_{ij}^{(n)}\|, \quad \text{where } a_{ij}^{(n)} = \binom{j-n}{i-1},$$

$$B^{(n)} = \|b_{ij}^{(n)}\|, \quad \text{where } b_{ij}^{(n)} = \binom{j-n-1}{i-n}.$$

If g is an element of an arbitrary abelian group G (here considered as a module over the integers), let us define $A^{(n)}g=\|a_{ij}^{(n)}g\|$, $B^{(n)}g=\|b_{ij}^{(n)}g\|$. It will be observed that these are precisely the Pascal matrices generated via (1) by an appropriate set of initial values whose only nonzero element is g. A direct consequence is: *the Pascal matrix M with entries in G that corresponds to the initial values $[g_{0i}; g_{ii}]$ can be considered as the sum of the infinite series* $\sum A^{(i)}g_{0i}+\sum B^{(i)}g_{ii}$. (Notice that each entry in M involves at most finitely many nonzero terms.)

3. Applications. The point of view of the preceding section enables us to solve (1) in terms of binomial coefficients and known initial values. For certain sets of initial values, (1) can be solved by other means and the resulting solutions written in a different form. Comparison of the two forms can lead to the discovery of some interesting combinatorial identities. In the following applications, all summations will range over all integers n, using the convention that $\binom{n}{k}=0$ if $n<k$, or if n or k is negative.

A. If each element of the set of initial values is the integer 1, the associated Pascal matrix $M=\|a_{ij}\|$ will satisfy $a_{ij}=\binom{j}{i}$. On the other hand, by the methods expounded here,

$$M = \sum A^{(n)} + \sum B^{(n)},$$

whence

$$a_{ij} = \sum a_{ij}^{(n)} + \sum b_{ij}^{(n)},$$

or

$$\binom{j}{i} = \sum \left[\binom{j-n}{i-1} + \binom{j-n-1}{i-n}\right].$$

This formula is not too difficult to derive by other means. The remaining examples should serve to illustrate the utility of the present method for the discovery of some lesser-known formulas.

B. Define

$$a_{ij} = \sum (-1)^n \binom{j+1-n}{i+1-n}.$$

This definition satisfies (1), and it is easy to calculate that $a_{0j}=j$ and that a_{jj} is 0 or 1 according as j is odd or even. Thus

$$\|a_{ij}\| = \sum nA^{(n)} + \sum B^{(2n)},$$

so

$$\sum (-1)^n \binom{j+1-n}{i+1-n} = \sum n \binom{j-n}{i-1} + \sum \binom{j-2n-1}{i-2n}.$$

C. More generally, if a_{ij} is any expression of the form

$$\sum c_n \binom{j+r_n}{i+s_n},$$

the method under discussion yields

$$a_{ij} = \sum \binom{j-n}{i-1} a_{0n} + \sum \binom{j-n-1}{i-n} a_{nn}.$$

The precise form of the resulting identity will, of course, depend on the form of the coefficients $[a_{0n}; a_{nn}]$.

DIFFERENT PROOFS OF DESARGUES' THEOREM

KAIDY TAN, Fukien Normal College, Foochow, Fukien, China

Desargues' Theorem is an exceedingly important one in modern geometry. In fact, it is sometimes regarded as the fundamental theorem of projective geometry. In this paper I give ten proofs; some are original, while others are taken from eminent works as shown in the references.

For the sake of brevity we adopt the following notations:

$A \cup B$	the line passes through the points A, B.
$A \cup B \cup C$	the three points A, B, C are collinear.
$AB \cap CD = E$	the lines AB, CD intersect at the point E.

or

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$A\cup B\cup C$	the three points A, B, C are collinear.
$AB\cap CD=E$	the lines AB, CD intersect at the point E.

$AB\cap CD\cap EF=P$	the three lines AB, CD, EF are concurrent at P.
$P\subset AB$	the point P lies on the line AB.
$AB\supset C$	the line AB contains the point C.
⫛	is homothetic with.
$\overset{O}{\overline{\wedge}}$	central perspective, or copolar; O being the center of perspective, or pole.
$\overset{l}{\overline{\wedge}}$	axially perspective, or coaxal; l being the axis of perspective.

DESARGUES' THEOREM. *If two triangles are centrally perspective, or copolar (i.e., all the lines joining corresponding vertices are concurrent), they are also axially perspective, or coaxal (i.e., the points of intersection of corresponding sides are collinear); and conversely, if two triangles are axially perspective, or coaxal, they are also centrally perspective, or copolar.*

We may prove the direct theorem first, i.e., if two triangles are centrally perspective (or copolar), they are also axially perspective (or coaxal).

In the Δs ABC, $A'B'C'$, if $AA'\cap BB'\cap CC'=O$, and $BC\cap B'C'=P$, $CA\cap C'A'=Q$, $AB\cap A'B'=R$, it is required to prove that $P\cup Q\cup R$.

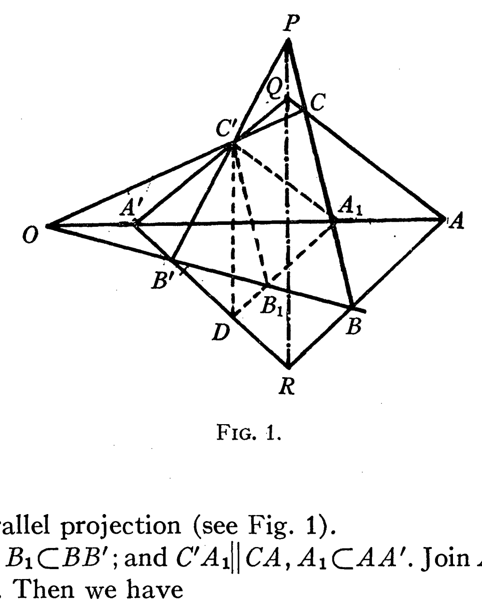

FIG. 1.

Proof 1. By parallel projection (see Fig. 1).

Draw $C'B_1 \| CB$, $B_1\subset BB'$; and $C'A_1\| CA$, $A_1\subset AA'$. Join A_1B_1; let $A_1B_1\cap A'B'=D$; and join DC'. Then we have

$$\frac{OB}{OB_1}=\frac{OC}{OC'}=\frac{OA}{OA_1}, \qquad \therefore\ B_1A_1\| BA.$$

Since

$$\frac{B'D}{B'R}=\frac{B'B_1}{B'B}=\frac{B'C'}{B'P}, \qquad \therefore\ C'D\| PR.$$

Again,

$$\frac{A'D}{A'R}=\frac{A'A_1}{A'A}=\frac{A'C'}{A'Q}, \qquad \therefore\ C'D\| QR$$

$$\therefore\ P\cup Q\cup R.$$

Proof 2. By properties of similar figures (see Fig. 2).

Let $\Delta ABC \underset{\lambda}{\overset{O}{=}} \Delta A'B'C'$, and $BC\cap B'C'=X$, $CA\cap C'A'=Y$, $AB\cap A'B'=Z$. Now we shall prove that $X\cup Y\cup Z$. Drop perpendiculars from the vertices of ABC ($A'B'C'$) upon the three sides of $\Delta A'B'C'$ (ΔABC). Label the feet of the perpendiculars P, P', Q, Q', R, R' as in Fig. 2. Also, from O, drop perpendiculars upon the three sides of $\Delta A'B'C'$ (ΔABC) the feet of the perpendiculars being V, W, U. Then, by similar figures,

$$\frac{BQ}{CR'}=\frac{BX}{CX}, \quad \frac{CR}{AP'}=\frac{CY}{AY}, \quad \frac{AP}{BQ'}=\frac{AZ}{BZ}.$$

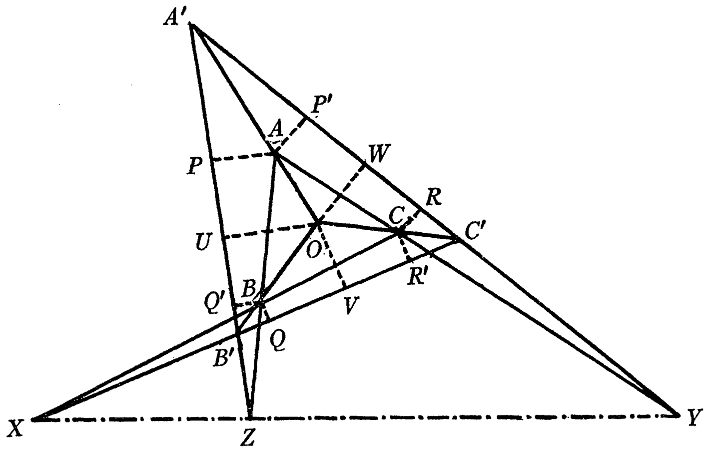

FIG. 2.

Also

$$\frac{AP}{AP'}=\frac{OU}{OW}, \quad \frac{BQ}{BQ'}=\frac{OV}{OU}, \quad \frac{CR}{CR'}=\frac{OW}{OV}.$$

$$\therefore\ \frac{BX}{CX}\cdot\frac{CY}{AY}\cdot\frac{AZ}{BZ}=\frac{BQ}{CR'}\cdot\frac{CR}{AP'}\cdot\frac{AP}{BQ'}=\frac{BQ}{BQ'}\cdot\frac{CR}{CR'}\cdot\frac{AP}{AP'}=\frac{OV}{OU}\cdot\frac{OW}{OV}\cdot\frac{OU}{OW}=1.$$

$\therefore$ the points X, Y, Z on the sides of ABC are collinear.

Proof 3. By the theory of homothecy.

LEMMA 1. *If a triangle has two sides parallel to two corresponding sides of another triangle and the lines joining corresponding vertices are concurrent, the two triangles are similar and similarly situated, or homothetic.*

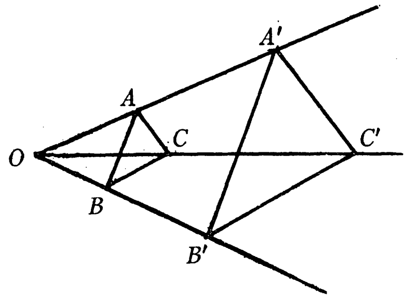

FIG. 3.

In other words, if two triangles are copolar, or centrally perspective (these two triangles are sometimes called homologous triangles), and two pairs of corresponding sides are parallel, then the third pair of corresponding sides are parallel also, and the two triangles are homothetic.

The proof is easy. Let $\Delta ABC \overset{O}{\barwedge} \Delta A'B'C'$, and $AB \| A'B'$, $AC \| A'C'$; then we shall prove that $BC \| B'C'$, and $\Delta ABC \phi \Delta A'B'C'$ (Fig. 3). Since

$$\frac{OB}{OB'} = \frac{OA}{OA'} = \frac{OC}{OC'}, \qquad \therefore\ BC \| B'C', \quad \text{and} \quad \Delta ABC \phi \Delta A'B'C'.$$

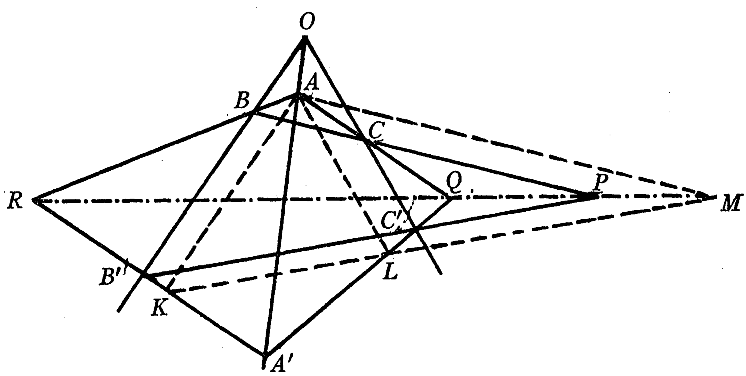

FIG. 4.

Now we begin to prove Desargues' Theorem as follows: Draw $AK \| BB'$, $K \subset A'B'$, and $AL \| CC'$, $L \subset A'C'$; join KL (Fig. 4). Now $\Delta OB'C' \overset{A'}{\barwedge} \Delta AKL$, and $OB' \| AK$, $OC' \| AL$, $\therefore B'C' \| KL$ (by Lemma 1). Next, draw $AM \| BP$, $M \subset KL$; then $\Delta ALM \phi \Delta CC'P$, $\therefore AC \cap LC' \cap MP = Q$. Similarly, $\Delta AKM \phi \Delta BB'P$, $\therefore AB \cap KB' \cap MP = R$. $\therefore P \cup Q \cup R$.

Proof 4. By the theory of transversal (see Fig. 5).

$A'B'R$ is a transversal of ΔOAB; hence by Menelaus' Theorem we have

$$\frac{OA'}{A'A} \cdot \frac{AR}{RB} \cdot \frac{BB'}{B'O} = -1.$$

Similarly, $C'B'P$ is a transversal of ΔOBC, so we have

$$\frac{OB'}{B'B}\cdot\frac{BP}{PC}\cdot\frac{CC'}{C'O}=-1;$$

and since $A'C'Q$ is a transversal of ΔOCA, we have

$$\frac{AA'}{A'O}\cdot\frac{OC'}{C'C}\cdot\frac{CQ}{QA}=-1.$$

Hence we obtain

$$\frac{AR}{RB}\cdot\frac{BP}{PC}\cdot\frac{CQ}{QA}=-1, \qquad \therefore\ P\cup Q\cup R.$$

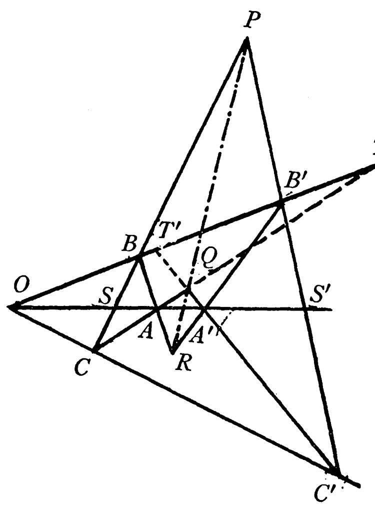

FIG. 5.

Proof 5. By the theory of cross-ratio (see Fig. 5).

Let OAA' intersect BC, $B'C'$ at S, S' respectively. Then $\{PBSC\}=\{PB'S'C'\}$, as both ranges lie on the pencil $O\{PBSC\}$. $\therefore A\{PBSC\}=A'\{PB'S'C'\}$, i.e., $A\{PROQ\}=A'\{PROQ\}$. These two equicross pencils, therefore, have a line OAA' in common. $\therefore P\cup Q\cup R$.

Proof 6. Alternate proof of above (see Fig. 5).

Let $CA\cap OB'=T$ and $C'A'\cap OB'=T'$. Then $\{CAQT\}=\{C'A'QT'\}$, $\therefore B\{CAQB'\}=B'\{C'A'QB\}$, and these equicross pencils have BB' in common. $\therefore P\cup Q\cup R$.

Proof 7. By projective method.

Let $\Delta ABC \overset{O}{\barwedge} A'B'C'$, $BC\cap B'C'=P$, $CA\cap C'A'=Q$ and $AB\cap A'B'=R$. Then we have to show that $QR\supset P$. Project QR to infinity. Then in the new figure, $A_1A_1'\cap B_1B_1'\cap C_1C_1'=O_1$ (here O_1, A_1, A_1', $\cdots$ denote the projections of

$O, A, A', \cdots$); also $A_1B_1 \| A_1'B_1'$, and $A_1C_1 \| A_1'C_1'$, $\therefore B_1C_1 \| B_1'C_1'$ (by Lemma 1), i.e., P_1 is at infinity, i.e., $P_1 \subset Q_1R_1$, i.e., $P_1 \cup Q_1 \cup R_1$. Hence in the original figure, $P \cup Q \cup R$.

Proof 8. Analytic method (see Fig. 6).

Take the two lines OAA', OBB' for the axes of Cartesian coordinates, and the coordinates of the vertices as $A \equiv (a, 0)$, $A' \equiv (a', 0)$, $B \equiv (0, b)$, $B' \equiv (0, b')$. $C \equiv (c, kc)$, and $C' \equiv (c', kc')$. Then the equation of AB is

$$\frac{x}{a} + \frac{y}{b} = 1; \quad \text{or} \quad bx + ay - ab = 0. \tag{1}$$

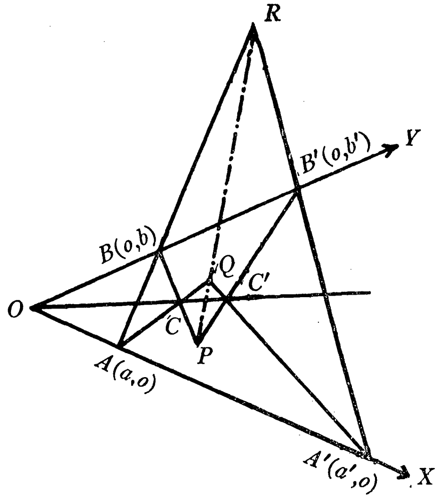

FIG. 6.

Similarly, $A'B'$ is the line

$$b'x + a'y - a'b' = 0. \tag{2}$$

Solving (1) and (2), we get the coordinates of R as

$$x_R = \frac{aa'(b - b')}{a'b - ab'}, \quad y_R = \frac{bb'(a' - a)}{a'b - ab'}.$$

Again, the equation of BC is

$$\begin{vmatrix} x & y & 1 \\ 0 & b & 1 \\ c & kc & 1 \end{vmatrix} = 0, \quad \text{or} \quad (b - kc)x + cy - bc = 0, \tag{3}$$

and $B'C'$ is the line

$$(b' - kc')x + c'y - b'c' = 0. \tag{4}$$

Solving (3), (4), we obtain the coordinates of P as

$$x_P = \frac{cc'(b-b')}{bc'-b'c}, \qquad y_P = \frac{bb'(c'-c)+kcc'(b-b')}{bc'-b'c}.$$

Finally, the equation of CA is

$$(5) \qquad \begin{vmatrix} x & y & 1 \\ a & 0 & 1 \\ c & kc & 1 \end{vmatrix} = 0, \quad \text{or} \quad kcx + (a-c)y - akc = 0$$

and $C'A'$ is the line

$$(6) \qquad kc'x + (a'-c')y - a'kc' = 0.$$

Solving (5), (6), we get the coordinates of Q as

$$x_Q = \frac{aa'(c-c')+cc'(a'-a)}{ca'-c'a}, \qquad y_Q = \frac{kcc'(a'-a)}{ca'-c'a}.$$

In order to prove $P \cup Q \cup R$, we may prove the following determinant to vanish.

$$D = \begin{vmatrix} aa'(b-b') & bb'(a'-a) & a'b-ab' \\ cc'(b-b') & bb'(c'-c)+kcc'(b-b') & bc'-b'c \\ aa'(c-c')+cc'(a'-a) & kcc'(a'-a) & ca'-c'a \end{vmatrix}$$

Multiplying the first column by k and subtracting it from the second column, we have

$$D = \begin{vmatrix} aa'(b-b') & bb'(a'-a)-kaa'(b-b') & a'b-ab' \\ cc'(b-b') & bb'(c'-c) & bc'-b'c \\ aa'(c-c')+cc'(a'-a) & -kaa'(c-c') & ca'-c'a \end{vmatrix}$$

$$= \frac{1}{(c-c')(a-a')(b-b')} \cdot \begin{vmatrix} aa'(b-b')(c-c') & bb'(a'-a)(c-c') - kaa'(b-b')(c-c') & (a'b-ab')(c-c') \\ cc'(b-b')(a-a') & bb'(c'-c)(a-a') & (bc'-b'c)(a-a') \\ aa'(c-c')(b-b') + cc'(a'-a)(b-b') & -kaa'(c-c')(b-b') & (ca'-c'a)(b-b') \end{vmatrix}.$$

Subtract the second and third row from the first row; then the elements of the first row in the new determinant are all zero; hence the determinant vanishes, $\therefore P \cup Q \cup R$.

Proof 9. Using homogeneous coordinates (see Fig. 7).

We can choose any point to be the unit point; let us therefore set $O \equiv (1, 1, 1)$, and let ABC be the triangle of reference with A as (1, 0, 0), B as (0, 1, 0), and C as (0, 0, 1). Then we can set $A' \equiv (1+\lambda, 1, 1)$, $B' \equiv (1, 1+\mu, 1)$, and $C' \equiv (1, 1, 1+\nu)$. Also AB is the line $z=0$, and the equation of the line $A'B'$ is

$$\begin{vmatrix} x & y & z \\ 1+\lambda & 1 & 1 \\ 1 & 1+\mu & 1 \end{vmatrix} = 0,$$

i.e., $\mu x + \lambda y - z(\lambda + \mu + \lambda\mu) = 0$.

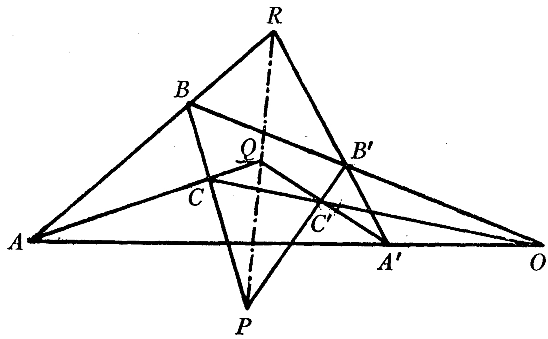

FIG. 7.

The two lines AB and $A'B'$ meet when $z=0$, i.e., when $\mu x + \lambda y = 0$, i.e., at the point $R(\lambda, -\mu, 0)$. Likewise P is $(0, \mu, -\nu)$ and Q is $(-\lambda, 0, \nu)$. It can now be easily verified that these three points lie on the line

$$\frac{x}{\lambda} + \frac{y}{\mu} + \frac{z}{\nu} = 0$$

and this proves the theorem. An alternate proof will be found in Loney's work, and we omit it here.

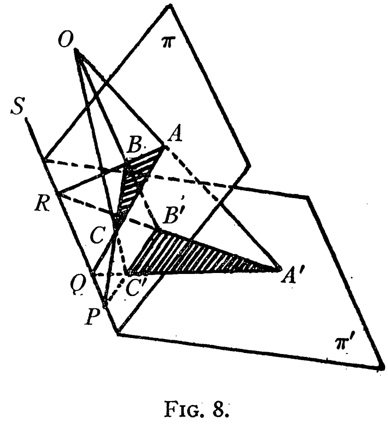

FIG. 8.

Proof 10. By use of the geometry of space.

Desargues' Theorem holds even when the two triangles are not in the same plane. When the two triangles are noncoplanar, the proof is simpler. As in Fig. 8, the ΔABC lies on the plane π, and the $\Delta A'B'C'$ lies on the plane π'; also $AA' \cap BB' \cap CC' = 0$. Since $AA' \cap BB' = 0$, the five points O, A, B, A', B' are coplanar. We have now three planes π, π' and $ABA'B'$ which intersect in pairs along the lines AB, $A'B'$, and S; so these three lines are concurrent at R, $\therefore R \subset S$. Similarly $Q \subset S$ and $P \subset S$, $\therefore P \cup Q \cup R$.

Note. The two triangles are centrally perspective; they are also called "perspective from a point," and the point O is called the "center of perspective" or the "center of homology." The two triangles are axially perspective; they are also called "perspective from a line," and the line PQR is called the "axis of perspective" or the "axis of homology."

When the two triangles are coplanar, we may use the proof of the noncoplanar case as above to prove it. Let $\Delta ABC \overset{O}{\barwedge} \Delta A'B'C'$, and the two triangles lie in a common plane π; also $BC \cap B'C' = P$, $CA \cap C'A' = Q$, $AB \cap A'B' = R$; we shall prove that $P \cup Q \cup R$ (see Fig. 9).

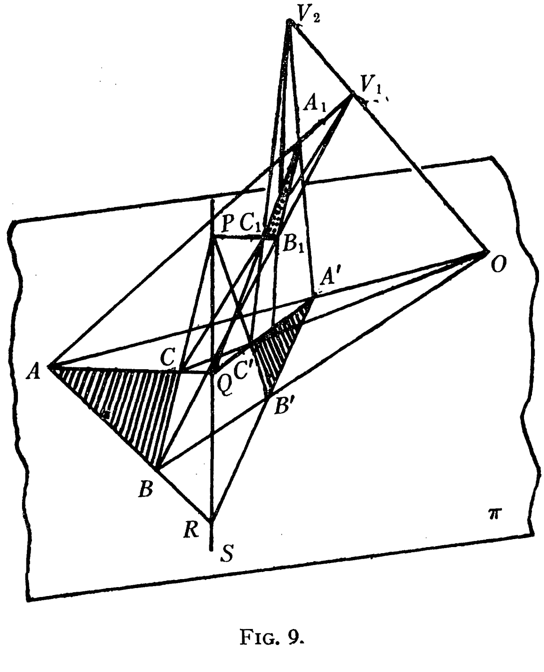

FIG. 9.

Draw any line through O not on the plane π and take two distinct points V_1 and V_2, neither coinciding with O, on this line. The lines V_1A, V_2A' are coplanar and intersect at a point A_1, for the four points A, A', V_1, V_2 lie on one plane, $\because AA' \cap V_1V_2 = O$. Similarly, $V_1B \cap V_2B' = B_1$, $V_1C \cap V_2C' = C_1$. The noncoplanar triangles $\Delta A_1B_1C_1 \overset{V_1}{\barwedge} \Delta ABC$; hence by the preceding case, they are perspective from the line $S = \pi \cap \pi_1$, where π_1 is the plane determined by the $\Delta A_1B_1C_1$, that is, B_1C_1 meets BC where the former cuts the plane π, etc.

Similarly, the noncoplanar triangles $\Delta A_1B_1C_1 \overset{V_2}{\overline{\wedge}} \Delta A'B'C'$, and they are perspective from the line S. That is, B_1C_1 meets $B'C'$ where the former cuts the plane π, etc.

Since B_1C_1 meets the plane π but once, $P=BC\cap B'C'$ is the point where $B_1C_1\cap\pi$, and furthermore, $P\subset S$. Similarly, $Q\subset S$, and $R\subset S$. $\therefore P\cup Q\cup R=S$.

In the following we shall prove the converse of Desargues' Theorem.

In $\Delta SABC$, $A'B'C'$, if $BC\cap B'C'=P$, $CA\cap C'A'=Q$, $AB\cap A'B'=R$, and $P\cup Q\cup R$; it is required to prove that $AA'\cap BB'\cap CC'=O$.

The converse theorem can be proved on the same lines as those employed for the direct theorem; or by a method of *reductio ad absurdum*. It can be also proved independently. Now we give a few methods as follows:

Proof 1. Apply the direct theorem.

Since $P\cup Q\cup R$, $\Delta BB'R \overset{P}{\overline{\wedge}} \Delta CC'Q$ (see Fig. 7); hence by the direct theorem they are axially perspective, and, therefore, $O=BB'\cap CC'$, $A=BR\cap QC$, and $A'=B'R\cap QC'$ are collinear. $\therefore AA'\cap BB'\cap CC'=O$.

Proof 2. By the theory of homothecy.

Draw $C'D\|QR$, $D\subset B'R$ and $C'B_1\|CB$, $B_1\subset BB'$; join DB_1, and let $A_1=DB_1\cap AA'$; join $C'A_1$ (see Fig. 1). Then

$$\frac{B'B_1}{B'B}=\frac{B'C'}{B'P}=\frac{B'D}{B'R}, \quad \therefore\ B_1D\|BR.$$

Hence

$$\frac{A'A_1}{A'A}=\frac{A'D}{A'R}=\frac{A'C'}{A'Q}, \quad \text{and} \quad C'A_1\|CA.$$

$\therefore \Delta A_1B_1C' \Phi \Delta ABC$, so $AA_1\cap BB_1\cap CC'=O$. $\therefore AA'\cap BB'\cap CC'=O$.

Proof 3. By the theory of transversal (see Fig. 7).

Since $P\cup Q\cup R$, let $AA'\cap CC'=O$; then we have to prove that $O\cup B\cup B'$. By Menelaus' Theorem, we have

$$\frac{CO}{OC'}\cdot\frac{C'A'}{A'Q}\cdot\frac{QA}{AC}=-1 \qquad (\because\ OAA' \text{ intersects } \Delta QCC')$$

$$\frac{C'B'}{B'P}\cdot\frac{PR}{RQ}\cdot\frac{QA'}{A'C'}=-1 \qquad (\because\ RA'B' \text{ intersects } \Delta QPC')$$

$$\frac{PB}{BC}\cdot\frac{CA}{AQ}\cdot\frac{QR}{RP}=-1 \qquad (\because\ RAB \text{ intersects } \Delta QPC)$$

$$\therefore\ \frac{CO}{OC'}\cdot\frac{C'B'}{B'P}\cdot\frac{PB}{BC}=-1.$$

$\therefore$ the points O, B, B' on the sides of $\Delta PCC'$ are collinear. Hence $AA'\cap CC'\cap BB'=O$.

Proof 4. By the theory of cross-ratio (see Fig. 5).

Since $P \cup Q \cup R$, $A\{PQRS\} = A'\{PQRS'\}$, where $S = OA \cap BC$, $S' = OA' \cap B'C'$. Hence $\{PCSB\} = \{PC'S'B'\}$. Since these two ranges have a common point P, $BB' \cap SS' \cap CC' = O$, i.e., $AA' \cap BB' \cap CC' = O$.

Proof 5. By projective method.

Project PQR to infinity. Then in the new figure $B_1C_1 \| B_1'C_1'$, $C_1A_1 \| C_1'A_1'$, and $A_1B_1 \| A_1'B_1'$, where A_1, A_1', $\cdots$ are the projections of A, A', $\cdots$ $\therefore \Delta A_1B_1C_1 ∯ A_1'B_1'C_1'$, and $A_1A_1' \cap B_1B_1' \cap C_1C_1' = O_1$. Hence in the original figure $AA' \cap BB' \cap CC' = O$.

Proof 6. Analytic method.

Let the equations of the three sides BC, CA, AB of the ΔABC be $u=0$, $v=0$, $w=0$ respectively, and the equation of the line PQR be $s=0$. Since $B'C'$ passes through the intersection of PQR and BC, its equation can be written as

$$s + lu = 0. \tag{7}$$

Similarly, the equations of $C'A'$, $A'B'$ are

$$s + mv = 0 \tag{8}$$

$$s + nw = 0 \tag{9}$$

respectively, where l, m, n are arbitrary constants. From (8) and (9), by subtraction, we have

$$mv = nw = 0 \tag{10}$$

This equation describes the line through A', the intersection of $C'A'$ and $A'B'$. But from the form of this equation we can easily observe that it represents the line through the intersection A of two lines $v=0$, $w=0$. Hence (10) represents the line AA'. Similarly,

$$nw - lu = 0 \tag{11}$$

$$lu - mv = 0 \tag{12}$$

express the lines BB', CC' respectively.

On adding the equations (10), (11), (12), their sum identically vanishes. So the straight lines represented by them meet in a point. $\therefore AA' \cap BB' \cap CC' = O$.

Proof. 7. By the geometry of space.

When the two triangles are noncoplanar (see Fig. 8), BC, $B'C'$, since they intersect at P, determine a plane α. Similarly CA, $C'A'$, intersecting at Q, and AB, $A'B'$, intersecting at R, determine planes β and γ. The three planes α, β, γ, intersect at a point $O = \alpha \cap \beta \cap \gamma$.

Now $AA' = \beta \cap \gamma$, for the points A and A' lie on both β and γ, $\therefore AA' \supset O$. Similarly $BB' = \gamma \cap \alpha$ and $CC' = \alpha \cap \beta$ pass through O. $\therefore AA' \cap BB' \cap CC' = O$.

When the two triangles are coplanar, from the direct theorem, Proof 10, the validity of the converse is assured by the principle of duality.

A proof without recourse to the principle of duality can also be developed along the lines suggested by the proof of the direct theorem as given in the former part Proof 10.

The other proofs are left to the readers; we shall give no more.

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BETWEEN T_2 AND T_3

B. T. SIMS, American University of Beirut

The purpose of this note is to point out a topological separation property which is between the classical properties T_2 and T_3 of Alexandroff and Hopf. This property has probably been investigated previously, but one finds no mention of it in current books on point-set topology. From its definition below, it is clear that this topological property implies T_2 and is implied by T_3.

DEFINITION. *A topological space* $\langle S, \mathfrak{J}\rangle$ *is a* $T_{5/2}$*-space if and only if for each pair of distinct points* $a, b \in S \exists U, V \in \mathfrak{J}$ *such that* $a \in U$, $b \in V$, *and* $\overline{U} \cap \overline{V} = \varnothing$.

Since open sets and their closures are preserved under homeomorphisms, it is clear that the property of being a $T_{5/2}$-space is topological. We give now two examples which show that $T_{5/2}$ is stronger than T_2 and weaker than T_3. The first example is due to Bing [1] and the second is due to Moore ([2], p. 65).

Example 1. Let $S = \{\langle x, y\rangle \mid x, y \text{ are rational}, y \geqq 0\}$. If $\langle a, b\rangle \in S$ and $\epsilon > 0$, the set $\{\langle r, 0\rangle \mid \text{ either } |r-(a+b/\sqrt{3})| < \epsilon \text{ or } |r-(a-b/\sqrt{3})| < \epsilon\} \cup \{\langle a, b\rangle\}$ is an ϵ-neighborhood of $\langle a, b\rangle$. The collection of all such neighborhoods is a basis for a connected topology $\mathfrak{J}$ on S which is T_2 and has the property that the intersection of the closures of any two neighborhoods is nonempty. Thus $\langle S\ \mathfrak{J}\rangle$ is not a $T_{5/2}$-space.

Example 2. Let $S = \{\langle x, y\rangle \mid x, y \text{ are real}, y \geqq 0\}$ and $L = \{\langle x, 0\rangle \mid x \text{ is real}\}$. Let d be the usual metric for E^2 and $S_d(p; r)$ the open d-sphere about p of radius $r > 0$. For each $p \in S$ and $r > 0$, we define a neighborhood $N_r(p)$ as follows: $N_r(p) = S_d(p;\ r) \cap S$ if $p \in S - L$; $N_r(p) = (S_d(p; r) \cap (S-L)) \cup \{p\}$ if $p \in L$. Clearly, the collection $\{N_r(p) \mid p \in S, r > 0\}$ is a basis for a topology $\mathfrak{J}$ on S. If p, q are distinct points of S, then $d(p, q) = 3r > 0$ for some $r > 0$. Since $r = 1/3d(p, q)$ implies $\overline{N_r(p)} \cap \overline{N_r(q)} = \varnothing$, $\langle S, \mathfrak{J}\rangle$ is a $T_{5/2}$-space. However, $\langle S, \mathfrak{J}\rangle$ is not regular. Consider the point $p = \langle 0, 0\rangle$ and the set $L - \{p\}$ which is closed. There do not along the lines suggested by the proof of the direct theorem as given in the former part Proof 10.

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DEFINITION. *A topological space $\langle S, \mathfrak{J}\rangle$ is a $T_{5/2}$-space if and only if for each pair of distinct points $a, b \in S \ni U, V \in \mathfrak{J}$ such that $a \in U$, $b \in V$, and $\overline{U} \cap \overline{V} = \varnothing$.*

Since open sets and their closures are preserved under homeomorphisms, it is clear that the property of being a $T_{5/2}$-space is topological. We give now two examples which show that $T_{5/2}$ is stronger than T_2 and weaker than T_3. The first example is due to Bing [1] and the second is due to Moore ([2], p. 65).

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One could make an investigation of the literature to see which of the theorems valid in T_3-spaces are also valid in the weaker $T_{5/2}$-spaces. However, we shall not pursue this course here. Instead, we conclude this note by showing that property $T_{5/2}$ is both hereditary and productive.

THEOREM 1. *If $\langle S, \mathfrak{I}\rangle$ is a $T_{5/2}$-space, then every subspace of $\langle S, \mathfrak{I}\rangle$ is also a $T_{5/2}$-space.*

Proof. Let $\emptyset\neq A\subset S$ and $p, q\in A$, $p\neq q$. Since $p, q\in S$, $\exists U, V\in\mathfrak{I}$ such that $p\in U$, $q\in V$ and $\overline{U}\cap\overline{V}=\emptyset$. Thus, $p\in A\cap U$, $q\in A\cap V$, and we have $(\overline{A\cap U}\cap A)\cap(\overline{A\cap V}\cap A)=\emptyset$ since $\overline{U}\cap\overline{V}=\emptyset$. Hence $\langle A, A\cap\mathfrak{I}\rangle$ is a $T_{5/2}$-space.

THEOREM 2. *If $\langle S_\alpha, \mathfrak{I}_\alpha\rangle$ is a $T_{5/2}$-space $\forall\alpha\in A$, then the product space $\langle\prod_A S_\alpha, \prod_A \mathfrak{I}_\alpha\rangle$ is also a $T_{5/2}$-space.*

Proof. Let $p, q\in\prod_A S_\alpha$, $p\neq q$. Thus $p_\alpha\neq q_\alpha$ for some $\alpha\in A$. Since $\langle S_\alpha, \mathfrak{I}_\alpha\rangle$ is a $T_{5/2}$-space, $\exists U_\alpha, V_\alpha\in\mathfrak{I}_\alpha$ such that $p_\alpha\in U_\alpha$, $q_\alpha\in V_\alpha$, and $\overline{U}_\alpha\cap\overline{V}_\alpha=\emptyset$. Thus, $\Pi_\alpha^{-1}(U_\alpha), \Pi_\alpha^{-1}(V_\alpha)\in\prod_A\mathfrak{I}_\alpha$ and $p\in\Pi_\alpha^{-1}(U_\alpha)$, $q\in\Pi_\alpha^{-1}(V_\alpha)$, where Π_α is the projection mapping of $\prod_A S_\alpha$ into $S_\alpha\forall\alpha\in A$. Since these projection mappings are continuous, we have $\overline{\Pi_\alpha^{-1}(U_\alpha)}\subset\Pi_\alpha^{-1}(\overline{U}_\alpha)$ and $\overline{\Pi_\alpha^{-1}(V_\alpha)}\subset\Pi_\alpha^{-1}(\overline{V}_\alpha)$. Also, $\Pi_\alpha^{-1}(\overline{U}_\alpha)\cap\Pi_\alpha^{-1}(\overline{V}_\alpha)=\emptyset$ since $\overline{U}_\alpha\cap\overline{V}_\alpha=\emptyset$. Thus $\overline{\Pi_\alpha^{-1}(U_\alpha)}\cap(\overline{\Pi_\alpha^{-1}(V_\alpha)}=\emptyset$ and $\langle\prod_A S_\alpha, \prod_A\mathfrak{I}_\alpha\rangle$ is a $T_{5/2}$-space.

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PALINDROMES BY ADDITION

CHARLES W. TRIGG, San Diego, California

Let N' be the integer obtained by writing the digits of the integer N in reverse order, and let $N+N'=S_1$, $S_1+S_1'=S_2$, $S_2+S_2'=S_3$, $\cdots$, $S_{k-1}+S_{k-1}'=S_k$. It has been conjectured, by whom I do not know, that for every N there is a k for which S_k is palindromic. Clearly, every N composed of digits <5 and every N with all digit pairs which are symmetrical to the middle having sums <10 will have a palindromic S_1. Thus, $S_1(2341)=3773$ and $S_1(28417)=99899$. Most other types of integers require greater k's. The N most frequently mentioned is 89, which after 24 successive reversals and additions produces $S_{24}(89)=881\ 32000\ 23188$. However, as will be shown, there are 249 integers $<10{,}000$ for which no palindrome appears for $k\leqq 100$. So, the conjecture is probably false.

All integers in which the corresponding digit pairs symmetrical to the middle have the same sums will produce the same palindrome. Thus 18, 27, 36, 45, 54, 63, 72, 81, and 90 all produce 99 in one operation, and may be said to *belong* exist two disjoint open subsets of S containing p and $L-\{p\}$, respectively. Thus, $\langle S, \mathfrak{J}\rangle$ is not a T_3-space.

One could make an investigation of the literature to see which of the theorems valid in T_3-spaces are also valid in the weaker $T_{5/2}$-spaces. However, we shall not pursue this course here. Instead, we conclude this note by showing that property $T_{5/2}$ is both hereditary and productive.

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Proof. Let $p, q \in \prod_A S_\alpha$, $p \neq q$. Thus $p_\alpha \neq q_\alpha$ for some $\alpha \in A$. Since $\langle S_\alpha, \mathfrak{J}_\alpha\rangle$ is a $T_{5/2}$-space, $\exists U_\alpha, V_\alpha \in \mathfrak{J}_\alpha$ such that $p_\alpha \in U_\alpha$, $q_\alpha \in V_\alpha$, and $\overline{U}_\alpha \cap \overline{V}_\alpha = \emptyset$. Thus, $\Pi_\alpha^{-1}(U_\alpha), \Pi_\alpha^{-1}(V_\alpha) \in \prod_A \mathfrak{J}_\alpha$ and $p \in \Pi_\alpha^{-1}(U_\alpha)$, $q \in \Pi_\alpha^{-1}(V_\alpha)$, where Π_α is the projection mapping of $\prod_A S_\alpha$ into $S_\alpha \forall \alpha \in A$. Since these projection mappings are continuous, we have $\overline{\Pi_\alpha^{-1}(U_\alpha)} \subset \Pi_\alpha^{-1}(\overline{U}_\alpha)$ and $\overline{\Pi_\alpha^{-1}(V_\alpha)} \subset \Pi_\alpha^{-1}(\overline{V}_\alpha)$. Also, $\Pi_\alpha^{-1}(\overline{U}_\alpha) \cap \Pi_\alpha^{-1}(\overline{V}_\alpha) = \emptyset$ since $\overline{U}_\alpha \cap \overline{V}_\alpha = \emptyset$. Thus $\overline{\Pi_\alpha^{-1}(U_\alpha)} \cap (\overline{\Pi_\alpha^{-1}(V_\alpha)} = \emptyset$ and $\langle \prod_A S_\alpha, \prod_A \mathfrak{J}_\alpha\rangle$ is a $T_{5/2}$-space.

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CHARLES W. TRIGG, San Diego, California

Let N' be the integer obtained by writing the digits of the integer N in reverse order, and let $N+N'=S_1$, $S_1+S_1'=S_2$, $S_2+S_2'=S_3$, $\cdots$, $S_{k-1}+S_{k-1}'=S_k$. It has been conjectured, by whom I do not know, that for every N there is a k for which S_k is palindromic. Clearly, every N composed of digits <5 and every N with all digit pairs which are symmetrical to the middle having sums <10 will have a palindromic S_1. Thus, $S_1(2341)=3773$ and $S_1(28417)=99899$. Most other types of integers require greater k's. The N most frequently mentioned is 89, which after 24 successive reversals and additions produces $S_{24}(89)=881\ 32000\ 23188$. However, as will be shown, there are 249 integers $<10{,}000$ for which no palindrome appears for $k \leqq 100$. So, the conjecture is probably false.

All integers in which the corresponding digit pairs symmetrical to the middle have the same sums will produce the same palindrome. Thus 18, 27, 36, 45, 54, 63, 72, 81, and 90 all produce 99 in one operation, and may be said to *belong* *to* 99. This set of nine integers may be represented compactly by the basic integer followed by the value of k in brackets, 18[1].

Likewise, 174[4] belong to 5115. That is, 174, 273, 372, 471, and 570 all produce 5115 after four reversals and additions. Similarly, 8699, 8789, 8879, 8969, 9698, 9788, 9878, 9968, or 8699[15] belong to 13 36977 96331.

It follows that in order to identify the palindromes to which the integers less than 10,000 belong, it is necessary to examine only 10 one-digit, 18 two-digit, 180 three-digit, and 342 four-digit numbers. These 550 basic numbers belong to 232 palindromes. More than half of the integers $<10{,}000$ are disposed of quickly, 2830 of them in one operation and 2900 others in two operations.

It may be observed that $S_1(a0b9)=1\,\overline{a-1}\,b\,\overline{b+1}\,\overline{a-1}$, $S_1(ab99)=1\,a\,b\,b\,\overline{a-1}$, and $S_2(a0b9)=a\,\overline{a+b}\,2b\,\overline{a+b}\,a=S_2(ab99)$, for $0<a\leqq 9$, $0\leqq b<9$. Hence integers of these two forms belong to the same palindrome with the same k, except for the cases $S_1(2009)=11011$, $S_2(2099)=22022=S_2(2009)$, and $S_1(2299)=12221$, $S_2(2029)=24442=S_2(2299)$. If $a+b>9$ or $b>4$, then $k>2$.

No integer $<10{,}000$ that produces a palindrome for $k\leqq 100$ requires more than 24 operations. However, there are eight integers 187[23]. They are part of the 98 integers, 89[24], 187[23], 869[22], 1297[21], 8039[20] and 8399[20], which belong to 881 32000 23188. To the only larger palindrome which appeared in this study, 1666 84884 86661, belong the four 6999[20] which are the only other integers requiring $k>19$.

The greatest number of integers $<10{,}000$ belonging to a single palindrome are the 219 integers, 99[6], 198[5], 1089[4], 3069[3], 3699[3], 7029[2], and 7299[2], which belong to 79497.

Integers yielding no palindromes during 100 operations. The 249 values of $N<10{,}000$ which do not produce palindromes for $k\leqq 100$ derive from 15 basic integers. The operations on these in turn fall into five addition patterns, since $S_5(196)=S_4(689)=S_3(1495)=S_2(4079)=S_2(4799)=13783$, $S_4(879)=S_3(1497)=S_2(8079)=S_2(8799)=96558$, $S_3(1997)=S_2(8089)=S_2(8899)=97768$, and $S_2(7059)=S_2(7599)=83127$. On pages 12–13 of volume 8(1938) of *Sphinx* (Bruxelles), D. H. Lehmer has stated that no palindromes occur for $S_k(196)$, $k\leqq 73$, nor for $S_k(1997)$, $k\leqq 76$.

The S_k, $k\geqq 100$, resulting from the five addition patterns are:

$S_{103}(196)=S_{102}(689)=S_{101}(1495)=S_{100}(4079)=S_{100}(4799)=$

865 71219 07263 72697 04998 66900 49696 28461 71802 18657

$S_{101}(1997)=S_{100}(8089)=S_{100}(8899)=$

3 71497 28748 57884 46665 77074 60874 66644 99757 47837 93172

$S_{102}(879)=S_{101}(1497)=S_{100}(8079)=S_{100}(8799)=$

55 99287 99079 28051 30078 14088 10305 17198 08979 39944

$S_{100}(7059)=S_{100}(7599)=$

1024 71276 12828 74024 17447 55843 82419 37829 30583 06430

$S_{100}(9999) =$

4912 42220 73662 72061 35411 21145 31712 71573 71222 42084

The number of integers leading to these five addition patterns are 113, 30, 72, 33, and 1, respectively, a total of 249. The sums encountered in obtaining the first two of these totals successively fit into the pattern of congruence to 7, 5, 1, 2, 4, 8, 7, · · · , modulo 9, while those in the third and fourth addition patterns are alternately congruent to 6 and 3 modulo 9. Those in the fifth addition pattern are congruent to zero modulo 99. These relationships were useful in checking the accuracy of the additions.

Occasionally in the addition patterns a value of S appeared which was almost palindromic, for example: $S_{16}(196) = 8971\ 00798$, $S_8(879) = 88\ 84788$, $S_6(7059) = 46\ 92864$, $S_{15}(1997) = 9\ 35232\ 32638$, $S_{46}(9999) = 38637\ 06276\ 57675\ 67270\ 63683$.

In none of the addition patterns was I able to see any regularity or periodicity that would justify the statement that no palindrome will occur for any $k > 100$. However, the occurrence of a palindrome in any of these addition patterns seems very unlikely.

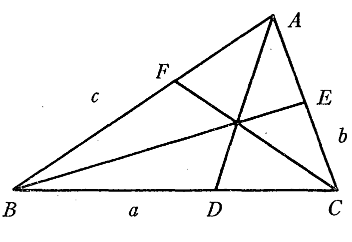
Added in proof. Don Wall of Princeton, N. J. informs me that he has developed these addition patterns to $S_{3556}(196)$, $S_{3717}(1997)$, and $S_{3765}(879)$ on an IBM 1401 computer to 1700-digit end results. No palindromes were encountered en route to these, nor to $S_{4147}(196)$. Neither were any repetitive patterns evident to him.

ON A NOTE BY LIN

CHARLES W. TRIGG, San Diego, California

The result in the second note by Tien-Hsung Lin [1] may be obtained more quickly by using a corollary of Ceva's Theorem [2]. Since the angle bisectors are cevians, we have

$$\frac{AI}{ID} = \frac{AF}{FB} + \frac{AE}{EC} = \frac{b}{a} + \frac{c}{a} = \frac{b+c}{a} > 1.$$



References

1. Tien-Hsung Lin, Notes, this Magazine, 38(1965), 158–159.
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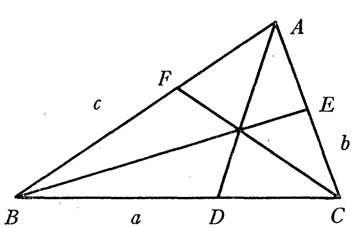

References

1. Tien-Hsung Lin, Notes, this Magazine, 38(1965), 158–159.
2. N. Altshiller-Court, College Geometry, Johnson Publishing, (1925), p. 131, Theorem 245.

ON PRIMES IN A.P.

A. M. VAIDYA, Gujarat University, Ahmedabad, India

A well known theorem of Dirichlet states that there are infinitely many primes in the arithmetic progression (A.P.)

$$a,\ a+d,\ a+2d,\ a+3d,\ \cdots \tag{1}$$

if $(a, d)=1$. (We shall assume that a and d are positive integers.) Since $a+d$ is a factor of $a+(1+am+dm)d$ for all integral values of m, it is clear that the A.P. (1) also contains infinitely many composite numbers. Thus it is interesting to ask the question: *How many successive terms of* (1) *can be primes*? In other words, if d is a given positive integer, what is the maximum value of $n=n(d)$, such that for some $q>p$, where p is the least prime not dividing d, the numbers

$$q,\ q+d,\ q+2d,\ \cdots,\ q+(n-1)d$$

are all primes? [The restriction $q>p$ is necessary to avoid a finite number of exceptions. For instance, with $d=6$, it can be easily shown that $\{5, 11, 17, 23, 29\}$ is the only set of 5 primes in A.P. with common difference 6. And, except for this set, there cannot be more than 4 primes in A.P. with common difference 6. Similarly $\{7, 157, 307, 457, 607, 757, 907\}$ is the only set of as many as 7 primes in A.P. with common difference 150; all other such sets cannot contain more than 6 primes. These facts easily follow from the main theorem of this paper.]

It is not difficult to prove that $n(2)=2=n(4)$, $n(6)=4$, $n(10)=2$, $n(30)=6$, etc. Generally, we prove the

THEOREM. $n(d)\leqq p-1$, *where p is the least prime not dividing d.*

Proof. For any integer $q>p$, consider the numbers

$$q,\ q+d,\ q+2d,\ \cdots,\ q+(p-1)d. \tag{2}$$

Now since $(d, p)=1$, the linear congruence $dx\equiv -q \pmod p$ has a solution. That is, for some integer m, $0\leqq m\leqq p-1$, p divides $q+md$ and, since $p<q\leqq q+md$, $q+md$ is not a prime. Thus there at least one composite number in (2). This proves the theorem.

One might make the following

Conjecture. $n(d)=p-1$, where p is the least prime not dividing d.

The conjecture is trivially true for all odd values of d. The author has verified the truth of the conjecture for all even values of d up to 210. For example, since $210=2\cdot3\cdot5\cdot7$, the conjecture would give $n(210)=11-1=10$. And we have the following set of 10 primes in A.P. with common difference 210:

$$\{199, 409, 619, 829, 1039, 1249, 1459, 1669, 1879, 2089\}.$$

One might also ask if for a given d, there are infinitely many sets of $n(d)$ primes in A.P. with common difference d. The special case $d=2$ of this question is well known and still unanswered. It asks whether there are infinitely many twin primes, i.e., primes p for which $p+2$ is also a prime.

Finally it may be mentioned that these results can be extended to rth power free integers where $r \geqq 2$. If $m = m(r, d)$ be the maximum possible number such that for some $q > p$ (p having the same meaning as before), the numbers

$$q, q + d, q + 2d, \cdots, q + (m - 1)d$$

are all rth power free, then it can be proved that $m(r, d) \leqq p^r - 1$, and it may be conjectured that $m(r, d) = p^r - 1$.

The author wishes to thank Professor A. R. Rao for suggesting the problem and the referee for his very valuable comments.

ANSWERS

A398. We note that $(x^2+1) - 2 = (x+1)(x-1)$. If $x+1$ and x^2+1 have a common divisor, then this divisor must also divide 2. Since x^2+1 and $x+1$ are both odd, this is not possible.

A399. Suppose rank $A = k$. Let $A = PBQ$ where P and Q are nonsingular and B has k ones down the main diagonal and zeros elsewhere. Then, $X = Q^{-1}BP^{-1}$ and satisfies the conditions of the problem.

A400. The only solution is $y=0$ since $D^n x^{2n} D^n \equiv x^n D^{2n} x^n$. This follows from $D^m x^m = x^m D^m + a_1 x^{m-1} + \cdots + a_m$, by Liebniz Theorem, $x^r D^r = xD(xD-1) \cdots (xD - r + 1)$. Since $xD - k_1$ commutes with $xD - k_2$, $D^m x^m$ commutes with $x^n D^n$ or $D^m x^{m+n} D^n \equiv x^n D^{m+n} x^m$.

A401. The intersection of the common external tangents of the two circles is the external center of similitude. A and B are homologous points. Homologous points are collinear with the center of similitude.

A402. Since during the change the Earth rotates on the average at half speed, it will lose one hour if Joshua begins at midnight.

A403. Assume $x = A^8$, $y = B^6$ and $z = C^5$. Then

$$(A/C)^{24} + (B/C)^{24} = C.$$

Let $A/C = M$ and $B/C = N$, giving the two parameter solution

$$x = M^8(M^{24} + N^{24})^8$$

$$y = N^6(M^{24} + N^{24})^6$$

and

$$z = (M^{24} + N^{24})^5.$$

Letting $M = N = 1$ yields the solution

$$x = 2^8, \quad y = 2^6, \quad z = 2^5.$$

(Quickies on page 54)

REMARKS ON THE FOUR COLOR PROBLEM; THE KEMPE CATASTROPHE

THOMAS L. SAATY, U. S. Arms Control and Disarmament Agency
and Conference Board of the Mathematical Sciences

1. Introduction. The original error discovered by Heawood in Kempe's attempt to prove the four color conjecture is often encountered by many of those who follow the inductive argument approach to the problem. Perhaps the nature of the difficulty is not well appreciated. An old Chinese proverb urges that "To know the road ahead, ask those coming back."

Here is one attempt at a proof which was recently pursued, along a fairly independent line of reasoning. In it Kempe's error is well discerned, and it might help shed some light on the difficulty encountered in using the inductive approach and encourage the examination of alternative methods, many of which are in the literature. In the process we also give a useful theorem regarding isomorphisms of graphs on six vertices with a minimum number of intersections.

2. Background. Instead of properly coloring the n regions of a map we shall be concerned with the proper coloring of the n vertices of its dual; a convenient fact which is possible due to planarity and not known to Kempe and Heawood since the equivalence of planarity and the existence of a dual were proved in the 1930's by H. Whitney. We apply the usual inductive argument on the $n-1$ vertex graph resulting from the removal of a vertex v with its five or less edges to all neighboring vertices. There is always such a vertex.

We note first that if the subgraph consisting of v with (in the worst case) all its five neighbors were complete, i.e., every pair of the six vertices is joined by an edge, the result would be a nonplanar graph and it is easy to show that it can have no less than three intersections. A complete graph on six vertices with three intersections exists and all complete graphs on six vertices with three intersections are isomorphic when the intersections are regarded as vertices. At least three edges must be removed from such a graph in order that it be planar, and hence a possible subgraph of our dual map. This follows from the fact that a sufficient condition for the nonplanarity of a graph is that the number of edges m satisfy $m > 3n-6$.

Once having decided which of the six vertices is v, it is possible to obtain a planar graph by removing three edges defining the intersections. The resulting graph has the largest number of edges that a simple planar graph on six vertices can have. A proper coloring of the $n-1$ subgraph (obtained after the removal of v and its five connecting edges) may leave the five neighbors of v colored with four colors with one possible duplication. Otherwise the solution is easily obtained. The problem is whether there are two pairs of vertices among v's five neighbors such that the vertices in each pair can never be joined by a chain. If this is the case, the vertices of each pair receive the same color and the fourth color is assigned to v.

Note that if such pairs exist in this maximally connected planar graph on six vertices, then one would hope that if some of the edges were absent in the dual map the corresponding vertices of a pair may be assumed to be connected by a chain of vertices which alternate in the color of these two vertices thus restoring a barrier between the vertices. If for example no such chain exists between the vertices x and y, then by reversing the color on one of the vertices, e.g., x, and all those vertices of a connected subgraph which includes x and consisting of vertices having the colors of x and y and their connecting edges, one may assign the same color to x and y, and thus hope to be left with an additional color to assign to v.

A look at the diagram below confirms this initial observation, but after proving the needed isomorphism we take a look at the subtlety involved.

3. An isomorphism. The proof of the following lemma is elementary.

LEMMA. *A complete graph on six vertices has at least three intersections.*

THEOREM. *Every complete graph on six vertices with a minimum number of intersections, when its intersections are regarded as vertices, is isomorphic to the graph of the following figure:*

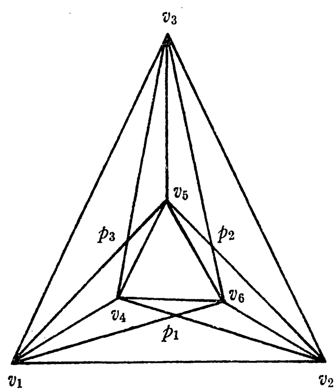

FIG. 1.

Proof. Let the original vertices and intersection points be denoted by v_i, $i=1, \cdots, 6$ and p_j, $j=1, 2, 3$, respectively.

(a) Each p_j is adjacent to precisely four v_i and no p_k, $k\neq j$ (by the nature of an intersection point p_j). Otherwise multiple intersections on an edge contradicts the minimality of the number of intersections.

(b) Each v_i is adjacent to precisely two p_j and three v_i. To prove this, note first that since the graph has 9 vertices and 21 edges (each intersection point divides an edge of the complete graph on six vertices into two edges of the graph with 9 vertices), by Euler's Theorem it divides the plane into 14 regions. The average number of edges bounding a region is three (each edge bounds two regions), and no region is bounded by fewer than three. Hence the regions are triangles. Between any two edges emanating from a given v_i and ending at two p_j's, there must be an edge ending at a v_m, $m\neq i$, since otherwise the two p_j's would be joined by an edge to complete a triangle. It follows that at most two p_j's are adjacent with any v_i. But the average number of p_j's adjacent with a v_i is two, hence precisely two p_j's and three v_i's meet any v.

(c) Of the 14 triangular regions, precisely two have no p_j's. To prove this, note that each p_j occurs in precisely four triangles (since it has degree 4) and no p's occur in the same triangle.

(d) The two triangles having no p_j's are vertex-disjoint. To prove this, note that they could not have precisely one common vertex, for then that vertex would meet four v_i's, which contradicts (b). Next, suppose they have two vertices and one edge in common, as in the following figure:

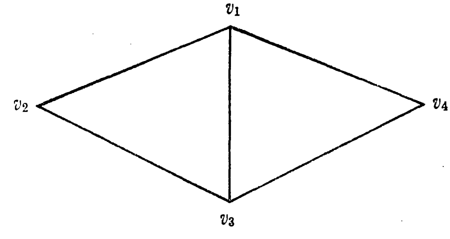

FIG. 2.

Assume that no further vertices and edges are inside the triangle determined by v_1, v_2, and v_3. Of the remaining two edges incident with v_1, which meet p_j's, one must lie in the outside region and one in the other triangle. Similarly at v_3. In fact, the graph must include a subgraph of the following form:

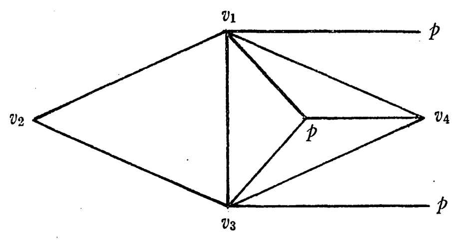

FIG. 3.

But then that p which is inside the triangle v_1, v_3, v_4 cannot be adjacent with a fourth v.

(e) If one of the two triangles in (d) is made the outside region, the graph includes the following subgraph:

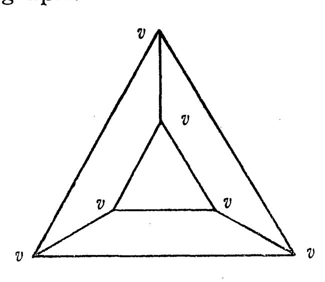

FIG. 4.

But then, there must be one p in each quadrilateral, in order for (a) to be satisfied. Thus the graph must have the stated structure. This completes the proof.

Note that six vertices may be joined to give a complete graph with more than three intersections. Some edges may have a multiplicity of intersections and these cases can also be treated separately. For example a single edge may have the following types of intersections (without completing the graph):

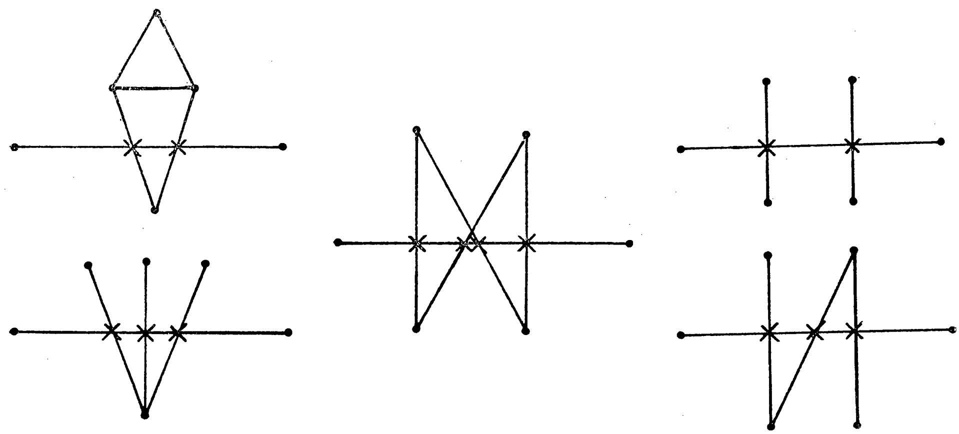

FIG. 5.

But here the connections when completed would show that as before at least two pairs of vertices exist in each of which one vertex is separated from the other. Thus one returns to settle first the apparently economical case given above.

4. Chains may cross, but edges may not. Using Fig. 1, we designate with v one of the vertices and fix its five connecting edges and then remove three edges in order to take out the intersections and obtain a planar graph. Let us suppose that the resulting figure is as follows:

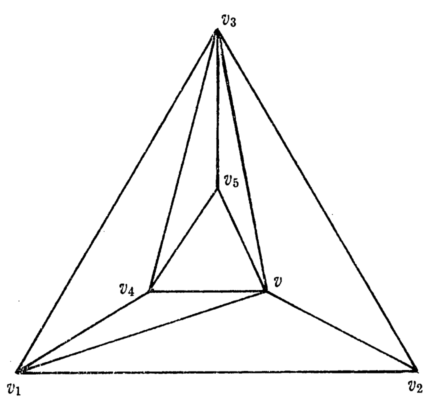

FIG. 6.

Let c_i be the color of v_i, $i=1, \cdots, 4$. Here we see that v_5 is separated from v_2 and from v_1. Also v_4 is separated from v_2. Thus we assign v_4 the color c_2 and *a fortiori* must assign v_5 the color c_1 and the remaining color c_4 is assigned to v.

Indeed our task would be finished if all the connections were edges. Suppose however, that in the absence of edges we have correspondingly a chain from v_1 to v_3 consisting of vertices alternately colored c_1 and c_3 and similarly a chain from v_4 to v_3 with vertices alternately colored c_4 and c_3. The chain v_1 to v_3 may cross the chain v_4 to v_3 as in the diagram:

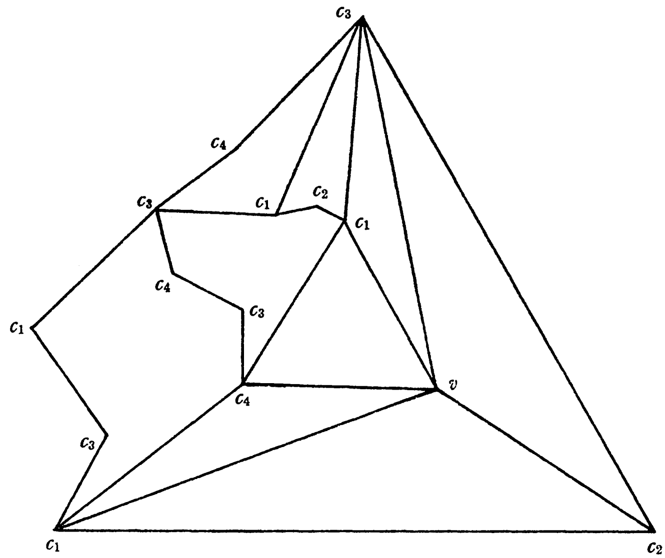

FIG. 7.

We know that v_5 must be assigned the color c_1. But suppose that the inductive argument results in the color c_2 assigned to v_5. Now we reverse the color on the connected subgraph of all vertices colored c_2 and c_1 which contains v_5. In this manner v_5 obtains the color c_1. However v_5 may have been connected by a chain of vertices colored with c_2 and c_1 to a vertex w colored with c_1 on the chain of vertices from v_1 to v_3. In that case reversing the colors on the above mentioned subgraph containing v_5 will change the color of w from c_1 to c_2.

Now v_4 is no longer separated from v_2 because they can be joined by a chain of vertices colored c_2 and c_4 which can meet the graph at a vertex colored c_4 on the exposed part of the chain (v_4, v_3) and then continuing on with vertices of alternate color c_2 and c_4 through the vertex w whose color has been changed to c_2 and finally terminating at v_4. (See Figure 8.)

Note that the chain from v_1 to v_3 has been broken and hence v_1 may be assigned the color c_3, but this does not yield an extra color to be assigned to v, nor is it possible to pursue the line of reasoning that in addition v_2 be assigned the color c_1.

This intermeshing of chains which is not possible for simple edges is none other than Kempe's Catastrophe on which much time is usually spent. It indicates the difficulty which the inductive argument presents. The difficulty is embodied in the basic requirement that whenever the color of one of the five vertices is altered to that of another, all vertices in a connected subgraph containing the vertex and consisting of vertices colored alternately with the same two colors as the two vertices under consideration be correspondingly reversed—the alternating backlash or feedback effect.

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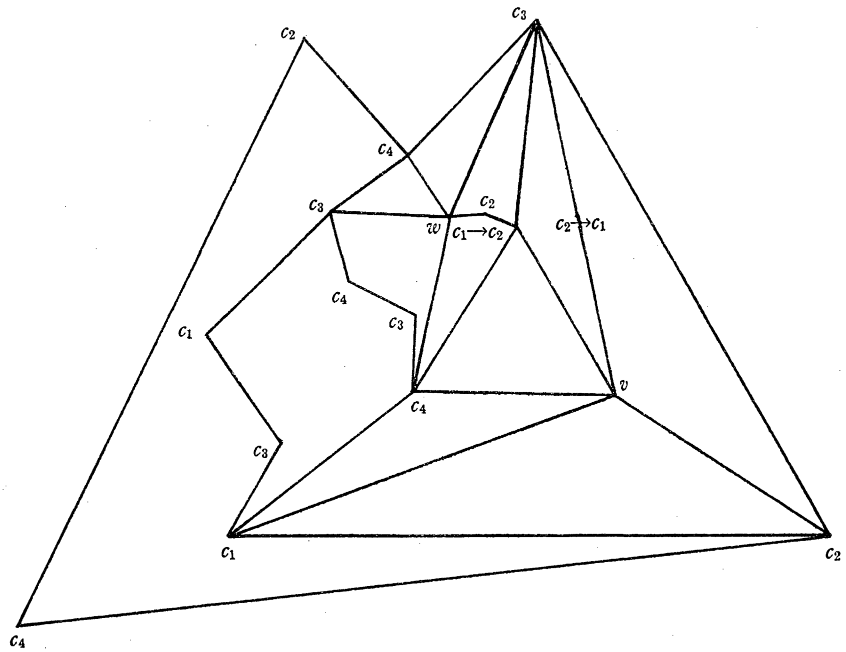

FIG. 8.

Reference

1. R. G. Busacker and T. L. Saaty, Finite Graphs and Networks; An Introduction with Applications, McGraw-Hill, New York, 1965. (See Chapter 4 for several pertinent references.)

A VARIATION OF THE BUFFON NEEDLE PROBLEM

R. L. DUNCAN, Lock Haven State College

We shall consider a problem in geometrical probability which is similar to the classical Buffon needle problem [1]. The solution of the Buffon needle problem will be obtained as a limiting case of the following problem.

Let rays with uniform angular spacing $2\pi/n$ be drawn from the point O on a board and suppose that a needle with midpoint M and of length $2L$ is thrown on the board. If $R=\overline{OM}$ and

$$(*) \qquad L \leqq R \sin \frac{\pi}{n},$$

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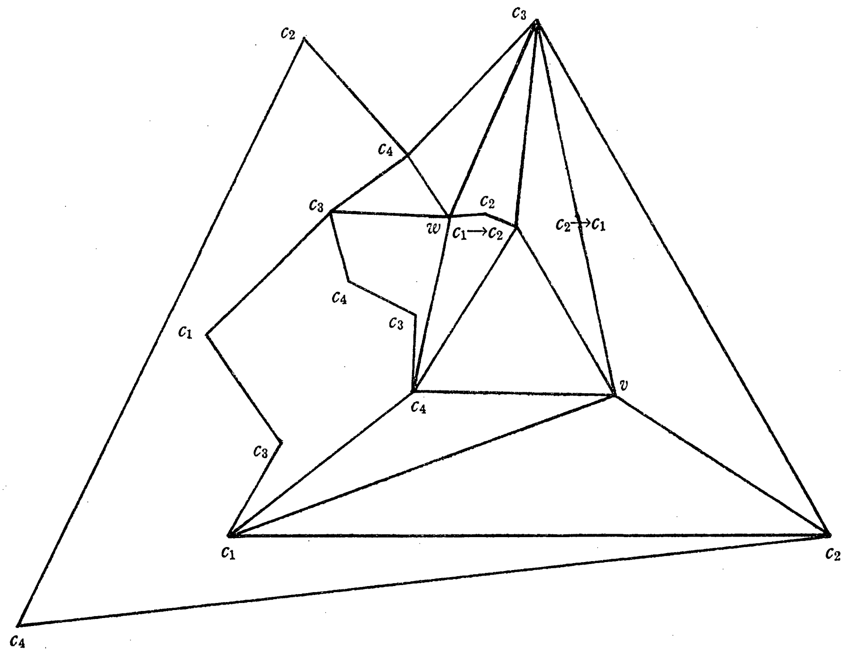

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Let rays with uniform angular spacing $2\pi/n$ be drawn from the point O on a board and suppose that a needle with midpoint M and of length $2L$ is thrown on the board. If $R=\overline{OM}$ and

$$L \leqq R \sin \frac{\pi}{n}, \tag{*}$$

what is the probability p that the needle will intersect one of the lines? Exact and approximate expressions for p are given by (1) and (2) below.

Let θ be the angle between OM and the nearest line and let ϕ be the smaller angle between OM and that half of the needle which intersects the interior of the angle θ. The needle will intersect one of the lines if and only if $R \sin \theta \leqq L \sin (\theta+\phi)$ and this is equivalent to

$$\theta \leqq \arctan\left(\frac{L \sin \phi}{R - L \cos \phi}\right).$$

We assume that θ and ϕ are independent and uniformly distributed on $0 \leqq \theta \leqq (\pi/n)$ and $0 \leqq \phi \leqq \pi$. Also, it is easily shown by differentiation that the maximum value of

$$\arctan\left(\frac{L \sin \phi}{R - L \cos \phi}\right)$$

on the interval $0 \leqq \phi \leqq \pi$ is $L/\sqrt{(R^2-L^2)}$ and by (*) this value does not exceed π/n. Hence the required probability is given by

$$p = \frac{n}{\pi^2}\int_0^\pi \arctan\left(\frac{L \sin \phi}{R - L \cos \phi}\right) d\phi, \tag{1}$$

since the integrand is nonnegative.

Since

$$\mu - \tfrac{1}{3}\mu^3 < \arctan \mu < \mu$$

for $|\mu| \leqq 1$, we have

$$p < \frac{n}{\pi^2}\int_0^\pi \frac{L \sin \phi d\phi}{R - L \cos \phi} = \frac{n}{\pi^2} \log \frac{R+L}{R-L}$$

for $n \geqq 4$. Also,

$$p > \frac{n}{\pi^2} \log \frac{R+L}{R-L} - \frac{n}{3\pi^2}\int_0^\pi \left(\frac{L \sin \phi}{R - L \cos \phi}\right)^3 d\phi$$

$$\geqq \frac{n}{\pi^2} \log \frac{R+L}{R-L} - \frac{n}{3\pi}\left(\frac{L^2}{R^2-L^2}\right)^{3/2} \geqq \frac{n}{\pi^2} \log \frac{R+L}{R-L} - \frac{n}{3\pi} \tan^3\left(\frac{\pi}{n}\right).$$

Hence,

$$p = \frac{n}{\pi^2} \log \frac{R+L}{R-L} + O\left(\frac{1}{n^2}\right). \tag{2}$$

If L is fixed and $n \to \infty$, then by (*), $R \to \infty$ and the configuration under consideration approaches that of a family of parallel lines with uniform spacing $d = 2R \sin \pi/n$ in any small sector at a distance R from O. If we let $L = \lambda R \sin \pi/n$, where L and λ are fixed and $0 < \lambda < 1$, then by (2) and L'Hospital's rule we have

$$p \sim \frac{n}{\pi^2} \log \frac{1 + \lambda \sin \frac{\pi}{n}}{1 - \lambda \sin \frac{\pi}{n}} \sim \frac{2\lambda}{\pi} \sim \frac{4L}{\pi d}$$

as $n, R \to \infty$.

This is just the solution of the classical Buffon needle problem for the case $2L \leqq d$.

Reference

1. J. V. Uspensky, Introduction to Mathematical Probability, McGraw-Hill, New York, 1937.

BOOK REVIEWS

EDITED BY DMITRI THORO, San Jose State College

Materials intended for review should be sent to: Dmitri Thoro, Department of Mathematics, San Jose State College, San Jose, California 95114.

Introduction to Computing. By T. E. Hull. Prentice-Hall, Inc. Englewood Cliffs, N. J., 1966 xi+212 pp. $6.95.

This little book covers a lot of territory. In just over 40 pages, the first four chapters introduce algorithms, flowcharts, machine language, programming techniques, and compilers and monitors. Chapter 1 treats the concept of algorithm informally and sets forth conventions to be followed in describing algorithms by flowcharts. A prevailing mathematical flavor is set forth early by regarding an algorithm as a transformation or mapping of an input data *domain* into an output *range*. The programming payoff of this viewpoint is not clear, but the notion is not taken up again until the final chapters dealing with Turing machines and finite state automata.

Chapters 2 and 3 introduce machine language programming by defining an order code vocabulary and other essential characteristics of a hypothetical computer. The technique of juxtaposing algorithms in flowchart form and the corresponding machine language helps to make the exposition clear. Even so, the concepts of address modification, looping, floating point arithmetic, open and closed subroutines, index registers, program linkages, and indirect addressing pass by in rapid fire order. The student with no previous computer contact may find it rough going.

Chapters 5 through 10 cover the general topic of FORTRAN IV programming and constitute better than one third of the book. In an evident attempt to provide a practical text, the author observes language and monitor conventions corresponding to IBSYS version 13 for the IBM 7094 computer and IBSYS version 9 for the IBM 7044 computer. For computing centers with these systems the book should have a special appeal. Throughout the FORTRAN programming chapters machine language code produced by the FORTRAN compiler is dis-

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Chapter 11 is a refreshing discussion of program planning and debugging. Hopefully, the student will have read these paragraphs before progressing very far through the programming exercises. The author discusses programming procedural matters as well as criteria by which good programs may be judged; namely, correctness, reliability, generality, convenience, and efficiency.

As regards a tendency to be brief, Chapters 12 through 14 on programming applications are no exception. Numerical, nonnumerical, and simulation applications are sketched. Numerical methods are illustrated by algorithms for solving linear equations and ordinary differential equations. Nonnumerical topics include discussions of algorithms for compiling simple algebraic expressions and formal differentiation. An interesting program for simulating a queue appears in Chapter 14.

Chapter 15 provides discussion of Turing machines, computable and noncomputable functions, finite state automata and the grammar of formal languages. The notion of algorithm is taken up again from a formal point of view. Generally, explanations are lucid and for the Turing machine an easy to follow example is described in detail. This chapter gives clear pointers to some deeper topics in computer science worthy of further study.

An outstanding characteristic of this book is the abundance of good exercises both of the pencil and paper and programming variety. Many of them are nontrivial and some amount to programming projects, rather than exercises only. A well organized summary of FORTRAN rules is included in appendices, making the book a convenient programming reference as well as a text. Somehow, though, it seems a shame to expend so much energy and competence to perpetuate FORTRAN between the covers of a hard bound book.

R. W. Cole, IBM, San Jose

The Significance of Mathematics. By Harriet F. Montague and Mabel D. Montgomery. Charles E. Merrill Books, Inc., Columbus, Ohio, 1963. xi+290 pp. $6.50.

The authors limit themselves to short, necessary explanatory notes in the preface and the introduction to the first chapter. They proceed immediately into very good, imaginative problems which are few enough so that the students are not overwhelmed by ten or twelve involved computational exercises after each section.

The content of the first chapters seems complicated. However, the students coming through better courses should find the vocabulary and processes completely understandable. The material is fast moving and may take an especially qualified teacher to fill in the details which are not covered. Foreseeing this cussed for selected situations. In particular, the machine language produced for logical IF statements and DO loops is explained in some detail. This should at least insure that the student understand why FORTRAN DO loops are so frustratingly restrictive. Essentially all FORTRAN IV programming features are covered, although some rather briefly. Included are subroutine linkage and parameter conventions, multiple subroutine entries, and the COMMON, DATA, and EQUIVALENCE statements.

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The content of the first chapters seems complicated. However, the students coming through better courses should find the vocabulary and processes completely understandable. The material is fast moving and may take an especially qualified teacher to fill in the details which are not covered. Foreseeing this possibility, however, the authors compiled comprehensive supplemental reading lists and included them at the end of every chapter.

The visual presentations, such as that for the sieve of Erathosthenes, are intriguing to behold. The section on unsolved problems is a good device for creating interest because the ones shown seem fairly easy on the surface and one or two students might very well think that it would be possible to come up with a new idea that could point toward a solution. The examples given throughout the book definitely tend to whet one's appetite and may set some unsuspecting music student (who takes a course in math to fill a liberal arts requirement) on the road to number theory before he remembers to tune his base fiddle.

Other subjects taken up in Chapter 2 tend to follow closely the new courses in the high schools concerning the use of other number bases than ten, the vocabulary of commutative, associative, etc. Certainly with so many students having been exposed to the more advanced vocabulary, a treatment of this kind seems highly appropriate. The explanations and proofs indicate a successful attempt to go into detail sparingly while still holding to acceptable rules of mathematics and making the chosen subject crystal clear. The language is concise and unwordy, and the complete sense of the theories is transmitted without resorting to unnecessary rigor and extraneous notation.

It was exceedingly refreshing to find—after the section on logic—a section called, "Applications of Symbolic Logic to the Design of Electrical Network." So often, great long discourses are set forth in complicated mathematical procedures with no mention of physical application. Many people need to see a "use" for what they learn and here a very practical example of an application is given much attention.

With the excellent diagrams illustrating clock numbers, rotation of squares, set relationships, and so on, those who may initially feel frightened of logic and higher mathematics are drawn in by the sheer simplicity and clarity of it. The consistent lack of tiresome verbiage serves to carry the reader swiftly from one subject to another wondering why it has always seemed so forbidding before.

Then, after becoming intrigued by these mathematical patterns, logical equivalences, and some "pure" mathematics, the historical background (brought in near the end of the book) is more meaningful. This section attracts the interest of even the least historically oriented person because it is interspersed with ways in which each new development made impact on bridge building, philosophy, art, religion, electrostatics, and quality control. It carries us through history country by country, from Babylonia to Boston; method by method, from counting by fives on fingers to programming computer machines; year by year, from 4000 BC to 1960; mathematician by mathematician, from al-Khowarizmi of Bagdad to Einstein; and theory by theory, from the Pythagorean Theorem to the theory of games. Again, this is done concisely, with a careful effort to avoid dull, unnecessary historical detail, and with numerous fascinating examples and illustrations.

In the final part the reader is brought up to date in probability and statistics with the appropriate vocabulary, the use of the Cartesian coordinate system, and Pascal's Triangle. The readers toss coins, play cards, roll dice, and thus learn to analyze data.

Chapter 14 goes into a sketchy introduction to some elementary concepts of the calculus which would have been better left out than hurried through in such a way as to make it look impossibly difficult to pursue. Again, however, the authors seem to tie the very difficult concepts of limit and continuity carefully, and without undue formal proofs, into a package of relationships between seemingly unrelated ideas.

JOAN L. PRESTON, San Jose, Calif.

An Introduction to Sets, Probability and Hypothesis Testing. By Howard F. Fehr, Lucas N. H. Bunt, and George Grossman. D. C. Heath, Boston, Massachusetts, 1964. v+245 pp. $3.88.

Basic notions related to sets, probability and hypothesis testing evolve explicitly within the six chapters of this textbook. Throughout the context there exists recommended phases for emerging mathematics programs that have been emphasized, and in some cases experimented with, by committees, writing groups, and individual teachers on both national and local levels.

A reviewer is made aware of the spiral development in employing concepts previously introduced to clarify and to enforce the learning of new ones. This fact becomes quite obvious in the probability sections where set and summation notations symbolize new and/or related ideas and in relationships found among combinations, the coefficients of the Binomial Theorem, and Pascal's Triangle. Along with the spiraling effects of content development emerge the challenging problems that comprise the exercises spaced at appropriate intervals.

Variations of notation appear to a reader familiar with other texts. Some examples are: (1) an open segment is represented as $\overset{\circ\ \ \circ}{\overline{AB}}$; (2) $(3, 5)\rightarrow 8$ implies the addition of the elements of the ordered pair; and (3) $A=\{(x, y): x^2+y^2=25\}$ rather than $A=\{(x, y)\,|\,x^2+y^2=25\}$.

The authors make the reader aware of the fact that ${}_nC_r$ may be written several ways, but then subscribe to Theorem 2–3 that ${}_nC_r=\binom{n}{r}$. It has been the reviewer's experience with both high school and college students that in order to avoid confusion, the symbolic form for combinations is best written as $C\binom{n}{r}$ so as to differentiate from the symbolic form for permutations of the type $P\binom{n}{r}$.

Set functions of an additive set function type generate interwoven relationships among notions of union, intersection, and disjointness in a unique manner. However, to elucidate the definition of an additive set function f that is first given, it might be helpful to present as the first example one that would not be an exception to the definition.

For the beginner in the study of statistical inference, the sections on hypothesis testing are written in a comprehendible manner. Nevertheless, consideration might be given to revealing relationships to the normal curve for clarification, especially for a high school student in the United States who has had no mathematical acquaintance with some elements of descriptive statistics.

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As one completes the reading of the expounded context, he finds it possible to subscribe to the fact that the organization of mathematical facts together with adequate illustrations enhance a clarity of comprehension not only for the teacher of mathematics, but also for a student of mathematics. It is the contention of the reviewer that exposure to the information contained on these well-printed pages should inspire high school seniors or college freshmen to further study related to probability and statistical inference.

REGINA H. GARB, Newark State College

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles City College, 855 North Vermont Avenue, Los Angeles, California 90029.

PROBLEMS

642. *Proposed by Maxey Brooke, Sweeny, Texas.*

I left Hooten-Holler and traveled west to Muletrack at 10 miles per hour. Sometime later, I returned to Hooten-Holler. My respective departure times were 8:00 a.m. and 10:00 a.m. local time. My respective arrival times were 9:25 a.m. and 11:35 a.m. local time. What was my probable means of transportation?

643. *Proposed by Richard L. Eisenman, U. S. Air Force Academy.*

Prove that the probability of a match (HH or TT) is 1/2 if and only if at least one of the coins is unbiased. Generalize to more than two coins.

644. *Proposed by Harlan L. Umansky, Union City, New Jersey.*

Find all the rectangles in which the area and the perimeter equal the same integer. Do the same for right triangles, equilateral triangles, and squares.

645. *Proposed by Esther Szekeres, University of Sydney, Australia.*

Given a convex quadrilateral, we drop from each vertex perpendiculars to the two sides not passing through it. Prove that if the sum of the lengths of these pairs of perpendiculars is the same for each vertex, the quadrilateral is a parallelogram.

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646. *Proposed by V. F. Ivanoff, San Carlos, California.*

Denoting the pairs of opposite vertices of a complete quadrilateral by A and A', B and B', C and C', respectively, prove that

$$\frac{AB\cdot AB'}{A'B\cdot A'B'} = \frac{AC\cdot AC'}{A'C\cdot A'C'}\,.$$

647. *Proposed by C. R. J. Singleton, Petersham, Surrey, England.*

Consider any nonnegative number which is a multiple of 3. Calculate the sum of the cubes of its digits. Calculate the sum of the cubes of the digits of this new number. Repeat this process indefinitely. Prove that any initial number will eventually generate the number 153.

648. *Proposed by Simeon Reich, Haifa, Israel.*

In the triangle $A_1A_2A_3$, let O_i be the midpoints of the sides and H_i the feet of the altitudes. Prove that

$$O_1H_2 + O_2H_3 + O_3H_1 = H_1O_2 + H_2O_3 + H_3O_1.$$

SOLUTIONS

Late Solutions

P. N. Bajaj, Western Reserve University: **619**; *Dermott A. Breault, Harvard Computing Center*: **615**; *John W. Milsom, Slippery Rock State College*: **614**; *S. Perlman, Wayne State University*: **608, 609, 613, 617, 619**; *Simeon Reich, Haifa, Israel*: **620**; *Kenneth A. Ribet, Brown University*: **615, 617**; *L. J. Upton, Port Credit, Ontario, Canada*: **613**.

Roots and Coefficients

621. [May, 1966] *Proposed by D. Rameshwar Rao, Osmania University, Secunderabad, India.*

If all three roots of the cubic

$$x^3 + px^2 + qx + r = 0$$

are real, then show that

(1) $p^2 \geqq 3q$

(2) at least one of the roots is less than or equal to

$$\frac{2(p^2 - 3q)^{1/2} - p}{3}\,.$$

I. *Solution by Vincent G. Sigillito, Silver Spring, Maryland.*

1. Part one will be solved in two ways:

(a) It is well known that the substitution $x = y - p/3$ reduces the cubic equation

$$x^3 + px^2 + qx + r = 0 \tag{1}$$

to

$$y^3 + ay + b = 0 \tag{2}$$

where

$$a = (1/3)\cdot(3q - p^2)$$
$$b = (1/27)\cdot(2p^3 - 9pq\ 27r).$$

Further (2), and hence (1), has three real roots iff

$$\frac{b^2}{4} + \frac{a^3}{27} \leqq 0. \tag{3}$$

For (3) to hold, we must have $a \leqq 0$ and this implies

$$p^2 \geqq 3q. \tag{4}$$

(b) Another derivation of (4) follows using the inequality

$$\frac{z^2 + w^2}{2} \geqq zw \tag{5}$$

which holds for all real z, w. Thus denoting the roots of (1) by r_1, r_2, r_3 it is easily shown that:

$$p = -(r_1 + r_2 + r_3)$$
$$q = r_1r_2 + r_1r_3 + r_2r_3$$
$$r = -r_1r_2r_3.$$

Then

$$\begin{aligned} p^2 &= r_1^2 + r_2^2 + r_3^2 + 2r_1r_3 + 2r_1r_3 + 2r_2r_3 \\ &= \frac{r_1^2 + r_2^2}{2} + \frac{r_1^2 + r_3^2}{2} + \frac{r_2^2 + r_3^2}{2} + 2r_1r_2 + 2r_1r_3 + 2r_2r_3 \\ &\geqq 3(r_1r_2 + r_1r_3 + r_2r_3) = 3q \end{aligned}$$

using (5). Equality holds in (4) iff $r_1 = r_2 = r_3$.

2. The bound

$$\frac{2(p^2 - 3q)^{1/2} - p}{3},$$

given by the proposer for the smallest root is easily obtained using well-known formulae for the solution of cubic equations. However in this situation, where all roots are real, a much better bound can be found. This bound follows from the fact that the smallest root, say r_1, is less than or equal to the x value where the first derivative of (1) is zero. Since (1) vanishes at

$$x = \frac{\pm(p^2 - 3q)^{1/2} - p}{3}$$

we see that

$$r_1 \leqq \frac{-(p^2 - 3q)^{1/2} - p}{3}.$$

It is easily shown that equality is attained in the above expression if the multiplicity of r_1 is greater than one.

II. *Solution by Hwa S. Hahn, State College, Pennsylvania.*

By translation $x = X - p/3$ we get

$$f(X) = X^3 - \frac{p^2 - 3q}{3} X + \frac{2p^3 - 9pq + 27r}{27} = 0.$$

Then

$$f'(X) = 3X^2 - \frac{p^2 - 3q}{3}.$$

By Rolle's Theorem, $f'(X)$ has two (possibly a double) real roots and so $p^2 \geqq 3q$ must hold. Since clearly $f(X)$ is convex downwards in $(\sqrt{(p^2 - 3q)/3}, \infty)$ and monotone increasing there, translating back to the original variable x we can now state that at least two of the roots is less than or equal to $\sqrt{(p^2 - 3q)/3} - p/3$. "At least two" can be replaced by "all" if $f(X)$ has a double root at this point.

Also solved by P. N. Bajaj, Western Reserve University (two solutions); Merrill Barneby, Wisconsin State University at La Crosse; Arthur Bolder, Brooklyn, New York; Sarah Brooks, Utica Free Academy, New York; Sandra Gossum, University of Tennessee, Martin Branch; H. R. Henshaw, Victoria, B. C., Canada; Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan; Edward F. Moylan, University of Wisconsin at Marshfield; Simeon Reich, Haifa, Israel; Kenneth A. Ribet, Brown University; Richard Riggs, Jersey City State College; Theron Rockhill, State University College, Brockport, New York; William Wernick, City University of New York; Louis R. Wirak, Lake Sumter Junior College, Florida; and the proposer.

S. Perlman, Wayne State University, found this problem in the Mathematical Gazette, October, 1965, *Page* 298.

A Rum Go

622. [May, 1966] *Proposed by Charles W. Trigg, San Diego, California.*

In various lands, there are discothèques with the name *Whisky A-Go-Go*. In the name of the Caribbean one,

$$RUM = AGO + GO,$$

each letter uniquely represents a digit in the scale of six. Rock "*n*" roll out the solution.

Solution by Marilyn R. Rodeen, San Francisco, California.

Since $R>A$ and $(25)_6+(25)_6=(54)_6$ with no carry-over, we need only consider the numbers from $(31)_6$ to $(54)_6$ for GO. Since A, R, G and O cannot be 0, and since each of the elements 0, 1, 2, 3, 4, 5 occurs in the cryptarithm, either U or M is 0.

$M=0$ if $GO=(43)_6$ or $GO=(53)_6$. But neither of these cases gives a satisfactory value for U. Hence, $U=0$, and this is true only if $G=3$ and there is no carry-over from $O+O$. This means that GO may be either $(31)_6$ or $(32)_6$.

Now the reader may easily verify that

$$\begin{array}{r} AGO \\ +\ GO \\ \hline RUM \end{array} \quad \text{is just another way of saying} \quad \begin{array}{r} 431 \\ +\ 31 \\ \hline 502 \end{array}$$

Also solved by Merrill Barnebey, Wisconsin State University at La Crosse; G. E. Bartel, Whitworth College, Washington; Sister Marion Beiter, Rosary Hill College, Buffalo, New York; C. R. Berndtson, MIT, Lincoln Laboratory; Arthur Bolder, Brooklyn, New York; Dermot A. Breault, Sylvania Electronics, Waltham, Massachusetts; Joseph Bohac, St. Louis, Missouri; Maxey Brooke, Sweeny, Texas; Sarah Brooks, Utica Free Academy, New York; R. J. Cormier, Northern Illinois University; Mark S. Fineman, Floral Park, New York; Frank Fishell, Montcalm Community College, Stanton, Michigan; Anton Glaser, Pennsylvania State University, Abington, Pennsylvania; Michael Goldberg, Washington, D. C.; Sandra Gossum, University of Tennessee, Martin Branch; Ernest R. Haylor, Bowling Green State University, Bowling Green, Ohio; J. A. H. Hunter, Toronto, Ontario, Canada; Jerome M. Katz, Brooklyn College; Jaques Labelle, Université de Montréal, Canada; Herbert R. Leifer, Pittsburgh, Pennsylvania; Edward F. Moylan, University of Wisconsin at Marshfield; William L. Mrozek, University of Michigan; C. C. Oursler, Southern Illinois University (Edwardsville); Prasert Na Nagara, Kasetsart University, Bangkok, Thailand; S. Perlman, Wayne State University; Stanley Rabinowitz, Far Rockaway, New York; Kenneth A. Ribet, Brown University; Richard Riggs, Jersey City State College; Donald R. Simpson, College, Alaska; Lewis Strumpf, State University College, Fredonia, New York; Paul Sugarman, Swampscott High School, Massachusetts; William Wernick, City University of New York; K. L. Yocom, South Dakota State University; and the proposer.

An Integer Form

623. [May, 1966] *Proposed by K. S. Williams, University of Toronto.*

Show that if a is any fixed integer of the form $4b^2+1$, then every integer can be put in the form $x^2+y^2-az^2$.

I. *Solution by Merrill Barnebey, Wisconsin State University at La Crosse.*

To put any integer in the form

$$x^2+y^2-az^2,$$

we first note that the squares of all odd integers are congruent to 1 (mod 8) and that the squares of all even integers are congruent either to 0 or 4 (mod 8). Next,

$$a = 4b^2 + 1 \equiv 1, 5 \qquad (\text{mod } 8),$$

and

$$az^2 \equiv 0, 1, 4, 5 \qquad (\text{mod } 8)$$

Therefore

$$x^2 + y^2 - az^2 \equiv 0, 1, 2, 3, 4, 5, 6, 7 \qquad (\text{mod } 8).$$

Thus if $a=4b^2+1$, any integer can be represented as $x^2+y^2-az^2$

II. *Solution by the proposer.*

As $a=4b^2+1$, the result follows immediately from the two identities:

$$2m + 1 \equiv (2bm)^2 + (m + 1)^2 - am^2$$

and

$$2m \equiv (m - 2b^2)^2 + [2bm - (4b^3 + 4b^2 + 2b + 1]^2 - a[m - (2b^2 + 2b + 1)]^2$$

Also solved by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan.

Five Spheres

624. [May, 1966] *Proposed by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan.*

Show that a sufficient condition for a sphere to exist which intersects each of four given spheres in a great circle is that the centers of the four given spheres be noncoplanar.

Solution by P. N. Bajaj, Western Reserve University.

Let the given spheres have equations

$$x^2 + y^2 + z^2 + 2u_ix + 2v_iy + 2w_iz + d_i = 0, \qquad i = 1, 2, 3, 4$$

referred to rectangular coordinates. Sphere $x^2+y^2+z^2+2Ux+2Vy+2Wz+D=0$ cuts these in the circles lying in the planes

$$2(U - u_i)x + 2(V - v_i)y + 2(W - w_i)z + (D - d_i) = 0, \qquad i = 1, 2, 3, 4.$$

If the circles of intersections are great circles, then

$$-2(U - u_i)u_i - 2(V - v_i)v_i - 2(W - w_i)w_i + (D - d_i) = 0, \qquad i = 1, 2, 3, 4$$

or

$$2Uu_i + 2Vv_i + 2Ww_i - D = 2u_i^2 + 2v_i^2 + 2w_i^2 - d_i, \qquad i = 1, 2, 3, 4.$$

A sufficient condition for these equations to determine U, V, W, D is

$$\det \begin{vmatrix} u_1 & v_1 & w_1 & 1 \\ u_2 & v_2 & w_2 & 1 \\ u_3 & v_3 & w_3 & 1 \\ u_4 & v_4 & w_4 & 1 \end{vmatrix} \neq 0$$

i.e., centers of the given spheres are nonplanar. Hence the result.

Also solved by Michael Goldberg, Washington, D. C.; Peter L. Langsjoen, Gustavus Adolphus College, St. Peter, Minnesota; Stanley Rabinowitz, Far Rockaway, New York; and the proposer.

A Limiting Probability

625. [May, 1966] *Proposed by Roy Feinman, Rutgers University.*

Consider n independent events. Let their probabilities of occurring be $(\frac{1}{2})^n$, i.e., $1/2, 1/4, \cdots, 1/2^n$. What is the limiting value of the probability that at least one of them occurs, as $n \to \infty$?

I. *Solution by James R. Kuttler and Nathan Rubinstein, Johns Hopkins University.*

The desired probability is one minus the probability that none occurs, which is given by

$$\prod_{k=1}^{n} [1 - (\tfrac{1}{2})^k].$$

Thus, to solve the problem we must compute an infinite product of the form

$$\prod_{k=1}^{\infty} (1 - q^k), \qquad 0 < q < 1.$$

This problem seems to have first been studied by Jakob Bernoulli and Euler (See Whittaker and Watson, *Modern Analysis*, Chapter XXI, §21.1), with work done on it by Jacobi and Gauss. The product is bound up with the theory of theta functions. In Exercise 10 of Chapter XXI, Whittaker and Watson give as due to Jacobi:

$$\prod_{k=1}^{\infty} (1 - q^k)^6 = 4\pi^{-3} q^{-1/4} k^{1/2} k'^2 K^3,$$

where

$$q = e^{-\pi K/K'}$$

and

$$K = \int_0^{\pi/2} (1 - k^2 \cos\theta)^{-1/2} d\theta_1$$

$$K' = \int_0^{\pi/2} (1 - k'^2 \cos\theta)^{-1/2} d\theta, \qquad k^2 + k'^2 = 1.$$

From formulas 16.38.5, 16.38.7, and 16.38.8 of *Handbook of Mathematical Functions*, Nat. Bur. Standards Appl. Math. Series 55, we can write this in terms of theta functions:

$$\prod_{k=1}^{\infty} (1 - q^k)^6 = \tfrac{1}{2} q^{-1/4} \theta_2(0, q)\theta_3(0, q)[\theta_4(0, q)]^4.$$

More practical for computation is the series representation of Euler, given in Bromwich, *An Introduction to the Theory of Infinite Series*, Chapter VI, Exercise 23:

$$\prod_{k=1}^{\infty} (1 - q^k) = \sum_{k=0}^{\infty} (-1)^k q^{k(3k+1)/2} + \sum_{k=1}^{\infty} (-1)^k q^{k(3k-1)/2}$$
$$= 1 - (q + q^2) + (q^5 + q^7) - (q^{12} + q^{15}) + (q^{26} + q^{28}) - \cdots,$$

the first nine terms of which, for $q = \frac{1}{2}$, yields

$$.288787841 < \frac{9463}{2^{15}} < \prod_{k=1}^{\infty} (1 - (\tfrac{1}{2})^k)$$
$$< \frac{77520901}{2^{28}} < .288787861.$$

The desired probability thus lies between .711212139 and .711212159.

Other solvers may have expected a simpler solution. Perhaps they referred to one of the many books containing the erroneous formula

$$\prod_{k=1}^{\infty} (1 + q^{2k}) = \frac{1}{1 - q},$$

which is apparently a misprint for the correct formula

$$\prod_{k=1}^{\infty} (1 + q^{2^k}) = \frac{1}{1 - q}.$$

II. *Solution by Kenneth A. Ribet, Brown University.*

The limiting probability is

$$1 - \prod_{i=1}^{\infty} [1 - (\tfrac{1}{2})^i]$$

If P is the value of the product, then

$$\log P = \sum_{i=1}^{\infty} \log [1 - (\tfrac{1}{2})^i]$$
$$= -\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left[\frac{1}{j} \left(\frac{1}{2} \right)^{ij} \right]$$

which may be reduced to

$$\log P = -\sum_{j=1}^{\infty} \frac{1}{j}\left(\frac{1}{2^j - 1}\right).$$

This series for log P converges much more rapidly than the infinite product for P. Using the series and a short FORTRAN program for a CDC 3600, the result produced a value for P of 0.2889. The desired probability, $1-P$ is .7111.

Also solved by Edward T. Frankel, U. S. Department of Health, Education and Welfare; Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan. Four incorrect solutions were received.

A Triangular Ratio

626. [May, 1966] *Proposed by Erwin Just and Norman Schaumberger, Bronx Community College.*

In triangle ABC let D, E and F be any points on sides AB, BC and CA respectively. Let G be the point of intersection of AE and DF. Prove that

$$\frac{DG}{GF} = \frac{AD}{AF}\cdot\frac{BE}{CE}\cdot\frac{AC}{AB}.$$

I. *Solution by Leon Bankoff, Los Angeles, California.*

$$\frac{BE}{EC} = \frac{\Delta ABE}{\Delta AEC} = \frac{AB\cdot AE \sin BAE}{AC\cdot AE \sin EAC} = \frac{AB \sin BAE}{AC \sin EAC}$$

or

$$\sin BAE/\sin EAC = BE\cdot AC/CE\cdot AB$$

Also

$$\frac{DG}{GF} = \frac{\Delta ADG}{\Delta AGF} = \frac{AD\cdot AG \sin BAE}{AF\cdot AG \sin GAF} = \frac{AD \sin BAE}{AF \sin GAF} = \frac{AD\cdot BE\cdot AC}{AF\cdot CE\cdot AB}$$

II. *Solution by Francine Abeles, Newark State College, New Jersey.*

Consider masses y and x placed at vertices B and C, respectively. To obtain DE, we consider it to arise from a split of the mass at A, i.e., some of the mass at A is combined with that at C to give the mass at F, while some is combined with that at B yielding the mass at D. We let the split mass at A be bx and ay, respectively. The center of mass of ayA and yB is D with mass $y(a+1)$; D divides AB in the ratio y/ay. Similarly, F divides AC in the ratio x/bx; E divides BC in the ratio x/y; G divides AE in ratio $x+y/ay+bx$ and DF in the ratio $x(b+1)/y(a+1)$. Hence

$$[x(b+1)/y(a+1)] = (y/x)(x/y)(x[b+1]/y[a+1])$$

or

$$DG/GF = (AD/AF)(BE/CE)(AC/AB).$$

We do not consider the degenerate cases: When D or F coincide with A, E coincides with B or C. However, if D coincides with B, we replace the split mass at A with z. Then

$$[x+z/y] = (x+z/x)(x/y),$$

i.e., $DG/GF=(AC/AF)(BE/CE)$. Similarly, when F is coincident with C, we have

$$DG/GC = (BE/CE)(AD/AB)$$

or

$$(x/y+z) = (x/y)(y/y+z).$$

Also solved by P. N. Bajaj, Western Reserve University; Arthur Bolder, Brooklyn, New York; Mannis Charosh, Brooklyn, New York; Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan; Lawrence V. Novak, Pennsylvania State University; Stanley Rabinowitz, Far Rockaway, New York; G. L. N. Rao, J. C. College, Jamshedpur, India; Simeon Reich, Haifa, Israel; Ronald R. Schryer, Orange Coast College, California; Agatha A. Sienicki, Immaculate College, Pennsylvania; Sister M. Stephanie Sloyan, Georgian Court College, New Jersey; Charles W. Trigg, San Diego, California; William Wernick, City University of New York; and the proposers.

Googolplex Congruence

627. [May, 1966] *Proposed by Harry W. Hickey, Arlington, Virginia.*

What is the remainder on division of googolplex by 7? ("Googolplex" $=10^g$ where $g=$ "googol" $=10^{100}$.)

Solution by Carl Hammer, UNIVAC, Wasington, D. C.

We shall prove more generally that

$$10^{10^\alpha} \equiv 4 \quad (\text{Mod } 7).$$

We note first that

$$10^\alpha - 4 \equiv 0 \quad (\text{Mod } 6)$$

since $10^\alpha-4$ is both even and divisible by 3 (its cross sum is $9\alpha-3$). Therefore, every power of 10 can be written as $10^\alpha=6n+4$. Further, the numbers (3, 2, 6, 4, 5, 1) form a complete residue system for successive powers of 10, Modulo 7. Therefore, every $(6n+4)$-th power of 10 will have a remainder of 4, Modulo 7. Finally, all these powers

$$10^{10^\alpha} = 10^{6n+4} \equiv 4 \quad (\text{Mod } 7),$$

including the case of $\alpha=100$ for which googolplex$\equiv 4$ (Mod 7).

Also solved by P. N. Bajaj, Western Reserve University; Bill Beckmann, Southwest DeKalb High School, Decatur, Georgia; Sister Marion Beiter, Rosary Hill College, Buffalo, New York; Joseph Bohac, St. Louis, Missouri; Arthur

Bolder, Brooklyn, New York; Maxey Brooke, Sweeny, Texas; Mannis Charosh, Brooklyn, New York; Raphael T. Coffman, Richland, Washington; R. J. Cormier, Northern Illinois University; Mark S. Fineman, Floral Park, New York; Herta T. Freitag, Hollins, Virginia; Sandra Gossum, University of Tennessee, Martin Branch; Anton Glaser, Pennsylvania State University, Abington, Pennsylvania; Hwa S. Hahn, State College, Pennsylvania; John R. Herndon, Stanford Research Institute; John E. Homer, Jr., St. Procopius College, Illinois; Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan; Jacques Labelle, Université de Montréal, Canada; David C. Lantz, Kutztown State College, Pennsylvania; Herbert R. Leifer, Pittsburgh, Pennsylvania; C. C. Oursler, Southern Illinois University (Edwardsville); Prasert Na Nagara, Kasetsart University, Bangkok, Thailand; Stanley Rabinowitz, Far Rockaway, New York; Kenneth A. Ribet, Brown University; Richard Riggs, Jersey City State College; Marilyn R. Rodeen, San Francisco, California; David L. Silverman, Hughes Aircraft Company, El Segundo, California; Paul Sugarman, Swampscott High School, Massachusetts; John Wessner, Melbourne, Florida; Louis R. Wirak, Lake-Sumter Junior College, Florida; K. L. Yocom, South Dakota State University; and the proposer.

Five incorrect solutions were received.

Comment on Problem 612

612. [January and September, 1966] *Proposed by M. B. McNeil, University of Missouri at Rolla.*

The integral

$$I_1 = \frac{1}{\pi^3}\int_0^\pi \int_0^\pi \int_0^\pi \frac{du\, dv\, dw}{1 - \cos u \cos v \cos w}$$

occurs in the study of ferromagnetism and in the study of lattice vibrations. Prove that

$$I_1 = (4\pi^3)^{-1}[\Gamma(1/4)]^4.$$

Comment by William D. Fryer, Cornell Aeronautical Laboratory, Buffalo, N. Y., and Murray S. Klamkin, Scientific Laboratory, Ford Motor Company, Dearborn, Michigan.

The sum

$$S = \sum_{n=0}^{\infty} \left\{\frac{1}{2^{2n}}\binom{2n}{n}\right\}^3$$

occurs in a combinatorial probability problem [1]. We evaluate the sum by two methods and obtain as a by-product some interesting equivalent expressions.

Since

$$(1) \qquad \binom{2n}{n} = \frac{2}{\pi}\int_0^{\pi/2} (2\cos\theta)^{2n} d\theta,$$

$$(2) \qquad S = \frac{4}{\pi^2}\sum_{n=0}^{\infty} \frac{1}{2^{6n}}\binom{2n}{n}\int_0^{\pi/2}\int_0^{\pi/2} (4\cos\theta\cos\psi)^{2n} d\theta d\psi.$$

By using

$$\sum_{0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{(1-4x)}},$$

(2) becomes

$$S = \frac{4}{\pi^2}\int_0^{\pi/2}\int_0^{\pi/2} \frac{d\theta d\phi}{\sqrt{(1-\cos^2\theta\cos^2\phi)}}$$
$$= \frac{4}{\pi^2}\int_0^{\pi/2}\int_0^{\pi/2} \frac{d\theta d\phi}{\sqrt{(1-\sin^2\theta\sin^2\phi)}}$$

or, in terms the complete elliptic function of the first kind,

$$S = \frac{4}{\pi^2}\int_0^{\pi/2} K(\sin\theta)d\theta$$
$$= \frac{4}{\pi^2}\int_0^{1} \frac{K(k)dk}{\sqrt{(1-k^2)}}.$$

The last integral is given in [2, p. 637] as

$$S = \frac{4}{\pi^2} K\left(\frac{1}{\sqrt{2}}\right)^2.$$

Identities leading to equivalent hypogeometric or gamma function forms may be found in the same reference (pp. 905, 909). Whence, also,

$$S = \frac{1}{4\pi^3}\Gamma(\tfrac{1}{4})^4 = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \tfrac{1}{2})^2 = 1.393203929685^+.$$

The sum S was also obtained as a by-product in establishing

$$(3) \qquad I = \frac{1}{\pi^3}\int_0^{\pi}\int_0^{\pi}\int_0^{\pi} \frac{dudvdw}{1-\cos u\cos v\cos w} = \frac{1}{4\pi^3}\Gamma(\tfrac{1}{4})^4$$

which is problem 612 in the *Mathematics Magazine* (January-February, 1966) due to M. B. McNeil.

References

1. W. Feller, An Introduction to Probability Theory and Its Applications, Wiley, New York, 1950

2. I. M. Ryshik and I. S. Gradstein, Tables of Series, Products and Integrals, Academic Press, New York, 1965.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q398. If x is an even integer, show that $x+1$ and x^2+1 are relatively prime.
[*Submitted by Norman Schaumberger and Judith Soriano*]

Q399. Show that the ring R_n of all $n\times n$ matrices over a field is regular, i.e., if $A\in R_n$, there exists $X\in R_n$ such that $A\times A=A$.
[*Submitted by Robert A. Melter*]

Q400. Find the general solution of the differential equation

$$\{D^n x^{2n} D^n - x^n D^{2n} x^n + \lambda^{2n-1}\} y = 0$$

[*Submitted by Murray S. Klamkin*]

Q401. Show that the vertex of a triangle A, the point of contact of the excircle relative to this vertex with the opposite side B, and the remote extremity of the diameter of the incircle perpendicular to this side are collinear.
[*Submitted by Charles W. Trigg*]

Q402. Suppose that Joshua had wished to change from Standard Time to Daylight Time, not advancing the clock, but by "stopping the sun" (See Joshua X:13): or in other words, by applying a constant deceleration to the rotation of the Earth, followed at once by a reacceleration at the same rate. When should he begin the process in order to complete it by 2 a.m.?
[*Submitted by H. S. Gould*]

Q403. Find positive integers x, y, and z such that $x^3+y^4=z^5$.
[*Submitted by David L. Silverman*]

(Answers on page 30)

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The Editorial Board acknowledges with thanks the services of the following mathematicians, not members of the Board, who have kindly assisted by evaluating papers submitted for publication in the MATHEMATICS MAGAZINE.

Henry Alder, C. B. Allendoerfer, J. W. Bishir, E. E. Burniston, W. B. Caton, John Cibulskis, H. S. M. Coxeter, Howard Eves, Michael Goldberg, R. L. Graham, Frank Harary, W. J. Harrington, V. Klee, Jack Levine, M. H. McAndrew, P. A. Nickel, Ivan Niven, Anthony Patricelli, B. L. Schwartz, R. A. Struble, W. T. Tutte, H. R. van der Vaart, W. B. Woolf, and J. D. Zund.

Finally it may be mentioned that these results can be extended to rth power free integers where $r \geqq 2$. If $m = m(r, d)$ be the maximum possible number such that for some $q > p$ (p having the same meaning as before), the numbers

$$q,\ q + d,\ q + 2d,\ \cdots,\ q + (m - 1)d$$

are all rth power free, then it can be proved that $m(r, d) \leqq p^r - 1$, and it may be conjectured that $m(r, d) = p^r - 1$.

The author wishes to thank Professor A. R. Rao for suggesting the problem and the referee for his very valuable comments.

ANSWERS

A398. We note that $(x^2+1)-2=(x+1)(x-1)$. If $x+1$ and x^2+1 have a common divisor, then this divisor must also divide 2. Since x^2+1 and $x+1$ are both odd, this is not possible.

A399. Suppose rank $A=k$. Let $A=PBQ$ where P and Q are nonsingular and B has k ones down the main diagonal and zeros elsewhere. Then, $X=Q^{-1}BP^{-1}$ and satisfies the conditions of the problem.

A400. The only solution is $y=0$ since $D^n x^{2n} D^n \equiv x^n D^{2n} x^n$. This follows from $D^m x^m = x^m D^m + a_1 x^{m-1} + \cdots + a_m$, by Liebniz Theorem, $x^r D^r = xD(xD-1)\cdots(xD-r+1)$. Since $xD-k_1$ commutes with $xD-k_2$, $D^m x^m$ commutes with $x^n D^n$ or $D^m x^{m+n} D^n \equiv x^n D^{m+n} x^m$.

A401. The intersection of the common external tangents of the two circles is the external center of similitude. A and B are homologous points. Homologous points are collinear with the center of similitude.

A402. Since during the change the Earth rotates on the average at half speed, it will lose one hour if Joshua begins at midnight.

A403. Assume $x=A^8$, $y=B^6$ and $z=C^5$. Then

$$(A/C)^{24} + (B/C)^{24} = C.$$

Let $A/C=M$ and $B/C=N$, giving the two parameter solution

$$x = M^8(M^{24} + N^{24})^8$$
$$y = N^6(M^{24} + N^{24})^6$$

and

$$z = (M^{24} + N^{24})^5.$$

Letting $M=N=1$ yields the solution

$$x = 2^8, \quad y = 2^6, \quad z = 2^5.$$

(Quickies on page 54)

NEW TEXTS AVAILABLE FOR WINTER AND SPRING CLASSES

EARL W. SWOKOWSKI

College Algebra

By Earl W. Swokowski, Marquette University. February, 1967, 368 pages, 6 × 9, tentative $7.50.

This text is designed for the first course in college algebra. It includes, in addition to the regular topics, material such as matrices, analytic geometry, permutations and combinations, and linear programming. A student who completes a course using this text should be well prepared to study the calculus. The book contains approximately 2000 exercises.

Table of Contents: 1. Fundamental Concepts of Algebra, 2. The Real Number System, 3. Algebraic Expressions, 4. Equations and Inequalities, 5. Functions and Graphs, 6. Exponential and Logarithmic Functions, 7. Systems of Equations and Inequalities, 8. Complex Numbers, 9. Polynomials, 10. Elements of Analytic Geometry, 11. Sequences, 12. Permutations, Combinations and the Binomial Theorem. Appendix. Answers to Odd-Numbered Problems. Index.

EARL W. SWOKOWSKI

Algebra And Trigonometry

By Earl W. Swokowski, Marquette University. March, 1967, 384 pages, 6 × 9, tentative $7.95.

Table of Contents: 1. Fundamental Concepts of Algebra, 2. The Real Number System, 3. Algebraic Expressions, 4. Equations and Inequalities, 5. Functions and Graphs, 6. Exponential and Logarithmic Functions, 7. The Trigonometric Functions, 8. Analytic Trigonometry, 9. Measurement of Triangles, 10. Systems of Equations and Inequalities, 11. Complex Numbers, 12. Polynomials, 13. Sequences. Appendix. Answers to Odd-Numbered Problems. Index.

ROBERT H. BREUSCH

Introduction To Calculus And Analytic Geometry

By Robert H. Breusch, Amherst College. January, 1967, 256 pages, 6 × 9, $8.50.

A clear and brief calculus of a single variable appropriate for students in the biological, managerial, and social sciences, and students of the liberal arts.

Table of Contents: Introduction, 1. The Basic Ideas of Analytic Geometry, 2. Functions and Limits, 3. Derivatives and Antiderivatives, 4. The Definite Integral, 5. The Differentiation of Algebraic Functions, and a Few Simple Integration Formulas, 6. Some Transcendental Functions, 7. Applications of Derivatives and Indefinite Integrals, 8. Conic Sections, 9. Differentials; Integration by Substitution, 10. Some Applications of Definite Integrals. Appendix I: Trigonometry, Appendix II: Exponents, Logarithms, Table of Trigonometric Functions. Answers. Index.

WILLIAM L. HART

Intermediate Mathematics With Applications

By William L. Hart, University of Minnesota. December, 1966, 357 pages plus 75 pages of tables, 6 × 9, $7.95.

This text presents a review of topics in elementary and intermediate algebra with real applications to business, economics and finance. The author is well-known for his clarity of presentation and his ability to communicate with students having a minimal mathematical background. The text includes linear programming, statistics, finite series, and introduces in the simplest form the ideas from the calculus involved in the study of marginal cost and revenue.

Table of Contents: 1. Introduction to Real Numbers, 2. Fractions and Integral Exponents, 3. Linear Statements in One Variable, 4. Decimals, Computation, and Percentage, 5. Simple Interest and Simple Discount, 6. Functions and Linear Equations in Two Variables, 7. Quadratic Equations in One Variable, 8. Radicals, General Exponents, and Logarithms, 9. Compound Interest, 10. Topics from Statistics, 11. Some Finite Series, 12. Annuities with Simple Data, 13. Inequalities and Linear Programming, 14. Marginal Cost and Marginal Review. Index.

RICHARD E. JOHNSON

Vector Algebra

By Richard E. Johnson, University of New Hampshire. Volume Four, 119 pages, 5⅝ × 8⅜ paper, $2.95.

Table of Contents: 1. Vector Spaces: Definition, Subspaces, Geometric vectors, Vector *n*-tuples. 2. Bases of a Vector Space: Independence, Bases, Dimension, Homomorphisms. 3. Inner Product Spaces: Inner products, Normal orthogonal bases. 4. Vector Algebras: Two-dimensional algebra, Three-dimensional algebra, Lines in space, Planes in space. Appendix: Fields, Coordinate systems, Unitary spaces, Historical note. Index.

LYLE E. MEHLENBACHER

Foundations Of Modern Mathematics

By Lyle E. Mehlenbacher, University of Detroit. Volume Six, November, 1966, 176 pages, 5⅝ × 8⅜ paper, $3.50.

Table of Contents: 1. The Mathematics Language, 2. The Calculus of Statements, 3. Universes, Sets, Boolean Algrebra, 4. Relations, Functions, and Graphs, 5. Rudimentary Algebras, 6. Geometries, Pure and Simple, 7. Analytic Geometry in the Cartesian Plane, 8. Analytic Geometry in 3-Space, 9. The Circular Functions. Appendix. Index.

MARGARET F. WILLERDING

Mathematical Concepts, A Historical Approach

By Margaret F. Willerding, San Diego State College. Volume Five, January, 1967, 112 pages, 5⅝ × 8⅜ paper, $2.95.

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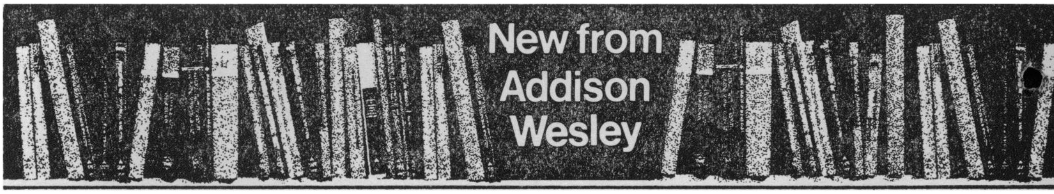

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